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PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

OCTOBER 1897.

I. *A Naval Range-Finder.*

By Prof. W. STROUD, *D.Sc.*, *M.A.**

HAVING been engaged for the past eight years in conjunction with Prof. Barr in the construction and perfecting of naval range-finders, it was natural that I should turn my attention to the possible application of telemetrical principles to physical measurement.

One very obvious application is to the determination of (1) the radius of curvature of a concave and especially of a convex surface, and (2) the focal length of a convex and especially of a concave lens. The problem of finding the position of the image of an object in a convex mirror or a concave lens is essentially a range-finding problem.

For the particular purpose in view the most suitable form of range-finder is a constant-range instrument, *i. e.* an instrument (whose optical parts are rigidly fixed) which can be translated along an optical bank and will indicate when an object or image is at the specified range. The mode of action of the instrument will thus be very like that of an ordinary short-focus telescope as commonly used on optical banks, but

* Read October 29, 1897.

a short-focus telescope is essentially bad as a range-finder, and that for the following reasons :—

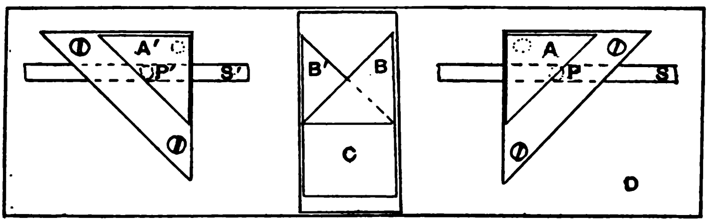
(1) As is well known, when using such a telescope it is advisable to cover up the middle portion of the objective so that only the extreme beams are used to form the image. In this way a crude range-finder is formed, which suffers from a radical defect, arising from the fact that the beams enter the eye in different directions, so that any alteration in the accommodation of the eye produces a duplication of the image. An attempt is made to avoid error from this cause by keeping the eye focussed all the time on cross-wires fixed in the focal plane of the eyepiece. (2) The short-focus telescope has not usually *base* enough for accurate telemetry (the maximum available base being the diameter of the objective), or if it should have base enough, the minimum range which it can measure is too great.

Looked at from the range-finding point of view the problem of optical spherometry and focometry is one of excessive simplicity. What is required is a range-finder whose prime reflectors can be fixed at any distance apart (to furnish the adjustable base) and at any angle to that base (to furnish the adjustable range).

One arrangement is shown in plan in fig. 1 and in perspective in fig. 2. A, A′ are two triangular reflecting prisms forming the prime reflectors supported on stands provided with levelling-screws. Slots S, S′ are provided in the frame-piece D through which pass stout pins P, P′ provided with clamping-screws below. B, B′ are fixed reflectors opposite A and A′ respectively. C is an extra reflector (made into one block with B and B′) so as to reflect the two beams upward to the eye. If now the prime reflectors be adjusted for a particular range, the two portions of the object viewed will appear in coincidence in the field of view furnished by C. If the two portions are not in coincidence they may be brought into coincidence by translating the range-finder to or from the object or image. The instrument works in fact just like a short-focus telescope, the only difference being that the correctness of the range is determined by observing whether or not the two portions of the object viewed are in alignment.

The instrument was used to determine the radius of curvature of a convex mirror by the ordinary method. It is clear that u should be made as large as possible so as to make v large.

Fig. 1.



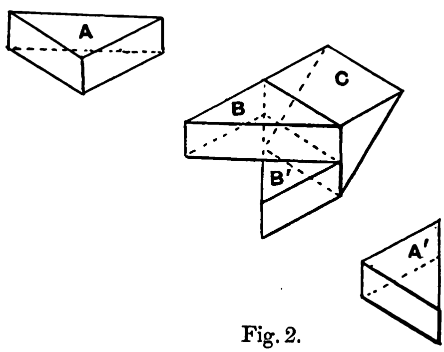


Fig. 2.

The object (a thin rod or slit well illuminated at the back) was placed at the extremity of the bank, and the mirror at the other extremity; the R.F. (range-finder) was brought up to read (1) the image of the rod, (2) the surface of the mirror, (3) the position of the object. Measurement (1) is first done because the R.F. should be adjusted on the image so as to have maximum available base-length and minimum range (the conditions for maximum accuracy). The portion of the mirror used was restricted to 10 cm. diameter, so as to make the measurements comparable with direct measurements made with our spherometer, which has its adjustable point 5 cm. distant from each of the fixed legs. The R.F. is then brought up to within a few cms. of the mirror and the prime reflectors are angled until the images appear nearly in coincidence, the prime reflectors are then rigidly fixed and

the R.F. is moved to and fro along the bank until exact coincidence has been obtained, when the reading on the bank is taken. If the base-length is 10 cm. and the image about 70 cm. away, the extreme variations cannot amount to 1 mm. and should not amount to more than $\frac{1}{2}$ mm. From fig. 1 it will be seen that the R.F. is so constructed that it does not itself obstruct the beams forming the two separate images provided its distance from the mirror is sufficient (say 8 or 10 cm.).

Measurement (2) is effected by suspending a weight by a very fine wire so that the wire rests in contact with the convex surface. Our mirror was silvered at the back, the wire resting against the front glass surface. Under these circumstances an image of the wire will be formed by reflexion in the surface. The R.F. is adjusted (*a*) on the wire, (*b*) on its image. If t is the thickness of the glass, $\frac{2t}{\mu}$ is the difference between these readings, from which t can be calculated with sufficient accuracy.

Measurement (3) is effected most conveniently by shifting the rod or slit a measured large distance (say 100 cm.) towards the mirror, so that the R.F. can be brought to bear upon it.

The following results were obtained :—

u.	v.	r.	r (corrected).
143·62	26·82	65·50	65·37
128·79	26·09	65·44	65 31
118·78	25·66	65·46	65·33
124·12	25·88	65·39	65·27
119·12	25·67	65·44	65·31

The last column gives the value for the radius corrected for aberration. The radius of the same mirror measured by our spherometer gave 65·31 and 65·37 cm. It should be remarked (1) that our spherometer has received a good deal of rough usage, (2) that the spherometer measures the curvature

of the external surface, while the telemeter was used to measure the curvature of the silvered surface. Still the results show that this telemetrical method is nearly as accurate as direct spherometry.

The constants of a concave lens were next determined. One face was obviously concave, the other looked nearly flat. The radius of the concave face was first got in this way:—the rod, suitably and strongly illuminated, was placed between the R.F. and the concave surface, and roughly adjusted so that it and its image were in close proximity near the centre of curvature. In other words u and v were made nearly equal and the value of each determined. In this case we are dealing with a real image and can bring our R.F. as near as we please to the image.

Now a little elementary calculation shows that if R be the range, dR the error in R, B the base, and $d\theta$ the error in angular estimation, then $d\theta = \frac{dR}{R} \cdot \frac{B}{R}$; but $\frac{B}{R}$ is practically fixed, as its maximum value $= \frac{\delta}{\rho}$, where δ, ρ are the diameter and radius of the surface respectively. Hence as $d\theta$ is fixed ($\doteqdot$ 1 minute of arc for snapshot naked-eye work and $\doteqdot \frac{1}{3}$ minute when great care is taken), dR will be a minimum when R is a minimum. To determine the range of a real image as accurately as possible, we must then take the minimum base and minimum range for maximum accuracy. In this case it may be advisable to place on the face of the prism a convex lens whose focal length is equal to the range required.

In the case of the lens used $\delta = 8$ cm. $\rho \doteqdot 16$ cm.; so that

$$dR = R \cdot \frac{R}{B} \cdot d\theta = 2R\, d\theta.$$

If, then, $d\theta = \frac{1}{3}$ minute $\doteqdot \frac{1}{10{,}000}$ and R $= 5$ cm., then

$$dR = \frac{1}{100} \text{ mm.}$$

In order to compete in accuracy with this a spherometer would require to indicate to $\frac{1}{3000}$ mm. In short, as a R.F. could be constructed to work at much shorter ranges than even 5 cm., there can be no question that for concave spherical surfaces the telemetrical method is more accurate than the spherometer.

In this way (though not with a very restricted base, for I did not realize at the time that the accuracy would be much increased by restricting the base and diminishing the range) the radius of curvature was found to be 15·81. A small spherometer gave 15·87 cm.—the lens face was too small for the large spherometer previously mentioned; the radius of the circle of contact of the small spherometer was only 1·5 cm., and as the instrument was home-made and only ordinary care had been bestowed on cutting the screw, I attribute the discrepancy to errors in the spherometer.

The second face when measured telemetrically turned out to be very slightly concave, the mean of several measurements giving a radius of 3300 cm.

The thickness of the lens can be readily got on the optical bank in this way:—The R.F. stand is so constructed that it can be turned through 180° about a vertical axis. A fine wire carrying a weight at its lower end is supported about the middle of the bank and readings of its position are taken with the R.F. from both sides.

The wire is now replaced by the lens and the operation repeated with the R.F., the readings for the one face being taken from the one side, the readings for the second face from the other.

The differences between the sets of readings for the wire and the corresponding sets for the lens gives the thickness required.

The principal points, the focal length, and the refractive index can now be obtained in the ordinary way.

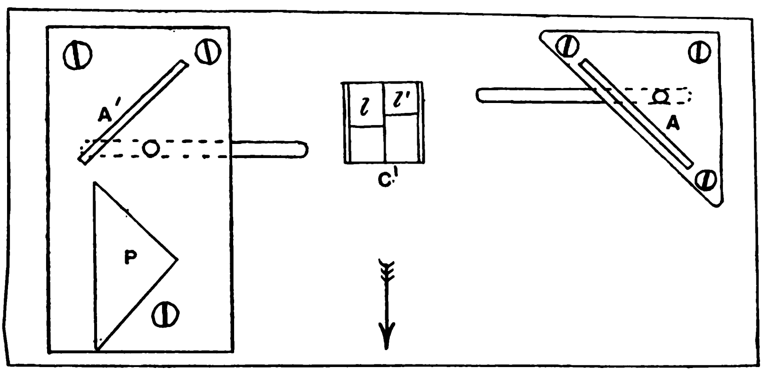
The following results were obtained for the focal length for sodium light, u and v being referred to the principal points.

u.	v.	f.	μ.
24·10	13·33	29·82	1·5276
31·40	15·31	29·87	1·5268
32·13	15·45	29·75	1·5287
35·21	16·13	29·77	1·5284
	Mean	29·80	1·5279

The results are remarkably good when it is remembered that all the observations are simple naked-eye observations without the assistance of any magnifying-power.

Fig. 3 shows in plan another form which the R.F. may take. A, A′ are, as before, the prime reflectors. The composite block of prisms B, B′, and C (figs. 1 and 2) is now replaced by a single right-angled polished speculum-prism C′ turned edge upwards, as shown in the plan. As it stands, this instrument would not be a range-finder at all, but an

Fig. 3.



azimuth indicator. A right-angled glass prism P is, however, mounted on the stage supporting the reflector A′, so that the beam of light is reversed right for left before falling on A′. This converts the instrument into a range-finder. The kind of appearance that is presented in the instrument before correct alignment has been attained is shown by the lines l, l', as seen in the prism C′. Here the object being observed is a line perpendicular to the plane of the figure at some distance away in the direction of the arrow. A moment's reflection will show that the R.F. is too near the object. To produce alignment we should require to move the instrument in the opposite direction to that of the arrow. The advantage which this arrangement presents over the preceding is that the speculum-prism C′ obstructs very little of the beams. Experiment shows that the accuracy obtainable with it is not appreciably different from that obtainable with the instrument first described.

Conclusions.

1. The telemetrical method described is not much inferior to a spherometer for measuring the radii of curvature of convex and approximately plane surfaces.

2. The method is superior in accuracy (though not of course in convenience) to the spherometer for measuring the radii of curvature of concave surfaces.

3. The method is available for determining all the geometrical or optical constants of either a convex or a concave lens on the optical bank alone.

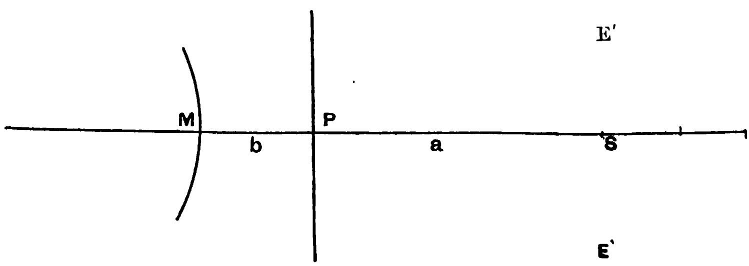
Appendix.

In connexion with this subject there remains to be described a simple apparatus for measuring optically the radius of curvature of a convex mirror or the focal length of a concave lens.

I venture to think that it will be found as useful in elementary teaching in other laboratories as it has been found at the Yorkshire College.

M is the convex mirror, P is a slip of silvered plate-glass (assumed plane), S is a pin well illuminated or narrow slit of light. The plate-glass P is arranged so that its upper edge is nearly on the horizontal axis of the mirror. Either S or P is shifted nearer to or further from M, until the image of S in P coincides with the image of S in M, whether viewed

Fig. 4.



from E or from E′. Under these circumstances $u=a+b$, $v=a-b$, whence r can be obtained. As the image formed by reflexion in M is smaller than that formed in P, it is advisable to use a very narrow slit at S.

To determine the focal length of a concave lens, the mirror is removed, the concave lens is placed so that its face touches the silvered back of P, and is half covered by it. A stout vertical rod is placed at some considerable distance behind the mirror on the prolongation of the axis of the lens, and the pin S is shifted to or from the mirror until the image of the pin seen by reflexion in the plane mirror coincides with the image of the rod as seen directly through the lens, whether we look from E or from E′. This gives the position of the virtual image of the rod, whence the focal length can be calculated.

The pin or rod can with advantage be replaced by a vertical straight edge well illuminated.

Both these methods will be found very satisfactory for elementary students. The apparatus is very simple, and the method seems to me instructive in illustrating the subject of parallax.

Discussion.

Mr. Barr drew attention to the gimbal arrangement and the three struts that keep the supporting rod central in the tube. To give some idea of the precision and scope of the range-finder, he observed that they were there using the equivalent of a 25-ft. "circle," and their measurements were comparable to the measurement of 20 secs. of angle, on such a circle. The instrument is handled by ordinary seamen, and stands rough usage on board ship for years without injury.

II. *Exhibition of a Focometer and Spherometer.* *By* Prof. STROUD.*

[Abstract.]

PROF. STROUD explained that in determining curvatures and focal lengths some telemetric method was necessary, and that, owing to want of parallelism of the beam and duplication of images, a short-focus telescope was always an inefficient telemeter. For the measurement of inaccessible lengths it was therefore better to use some simple form of "range-finder." Such an apparatus could be made with a set of small mirrors, arranged in such a manner as to direct two images of the distant object into an eyepiece, with a fixed prism in the path of one of the incident beams. By sliding this instrument along the optical bench, one position could always be found at which the two images, as seen through the eyepiece, were in coincidence. He also described a method for determining curvature by interposing a plate of plane glass between the curved mirror and a source of light.

III. *Experimental Exhibitions.* *By* Mr. ACKERMANN.*

[Abstract.]

MR. ACKERMANN exhibited two experiments:—(1) The blowing-out of a candle-flame by the air from a deflating soap-bubble. The bubble was blown at the mouth of an inverted beaker by breathing into a hole cut out at the top. This hole was then presented to the flame, and the flame was immediately quenched. But if the bubble was blown from ordinary air with bellows, the flame was merely deflected without being extinguished. (2) It was shown that a miniature boat, provided with a false stern, consisting of a linen diaphragm, could be propelled by filling the hollow stern-space with ether, or with some liquid similarly miscible with water. The motion is due to the continuous release of surface-tension behind the boat.

* October 29, 1897.

Prof. BOYS said that when he tried, some years ago, to blow out a candle with a soap-bubble filled with common air, he found the operation very difficult—so difficult that, having once succeeded, he never repeated the attempt. It had not occurred to him, as it had to Mr. Ackermann, that the CO_2 present in the breath played a part in the quenching. With regard to the second experiment, he had seen a small boat propelled by dissolving camphor astern, but he thought the use of a liquid for that purpose was a novelty.

IV. *On the Isothermals of Ether.*

By J. ROSE-INNES, *M.A., B.Sc.**

IN a paper "On the Isothermals of Isopentane," read before the Physical Society last May†, I gave an account of some results obtained by investigating Prof. Young's experimental work on Isopentane. The main conclusions arrived at were as follows:—

(i.) The study of a diagram constructed by plotting $\frac{1}{av^2}$ against $v^{-\frac{1}{3}}$, where a is the internal pressure and v is the volume of a gram, suggests that there is a discontinuity in the slope of $\frac{1}{av^2}$. Even if there is not discontinuity in the exact sense of the word, there is an extremely rapid change of behaviour, amounting practically to the same thing.

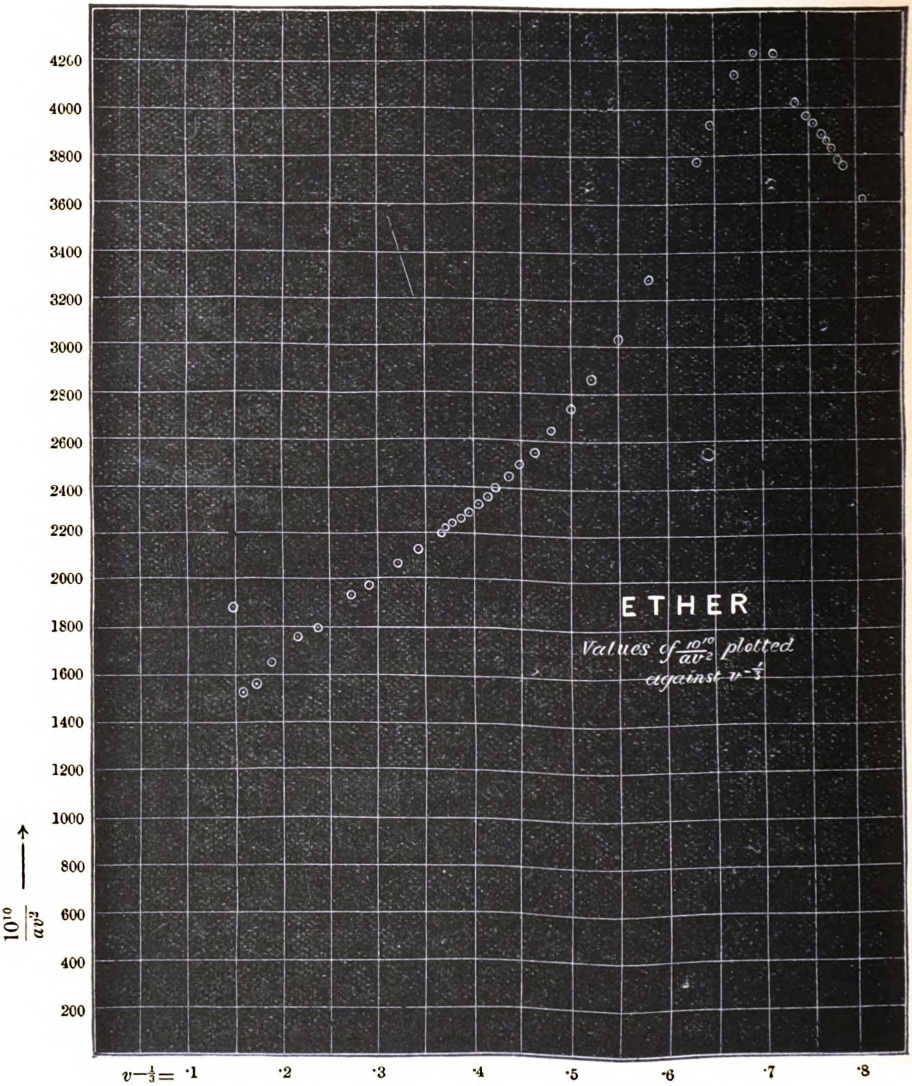
(ii.) The temperature for which the pressure is accurately given by the laws of a perfect gas at a given volume, remains practically constant for all large volumes, until we approach the neighbourhood of the critical volume. At the critical volume this temperature has diminished somewhat from its value for large volumes, but the diminution is only slight.

These conclusions were embodied in a formula giving the pressure in terms of the temperature and volume, and a comparison between calculation and experiment was effected by means of a diagram. It is not suggested that this formula

* Read November 12, 1897.

† Proc. Phys. Soc. xv. p. 126.

is incapable of further improvement, but the close correspondence between calculation and experiment seems to show that



the main conclusions on which the formula was founded are correct.

A natural extension of the above investigation is to try how far these general conclusions are true of some other substance. For this purpose I resolved to employ the experimental results obtained by Profs. Ramsay and Young with ether, as it appears that the linear law connecting temperature and pressure at constant volume holds accurately in this case. [Phil. Trans. vol. 178 A, pp. 57–93; and Phil. Mag. vol. xxiii. pp. 435–458.]

Profs. Ramsay and Young have given the values of a for a large number of volumes (Phil. Mag. vol. xxiii. p. 441), so that it was easy to calculate the values of $\frac{1}{av^2}$, and to plot them against $v^{-\frac{1}{2}}$. The resulting diagram is given on p. 12, and it suggests very strongly that there is a discontinuity of slope at about vol. 3·3. In this respect, accordingly, the ether results show a striking likeness to those of isopentane, the discontinuity in the case of the last substance occurring at about the same volume.

It was shown in my former paper that when Ramsay and Young's linear law holds, there is one and only one temperature for each volume at which the pressure is given by the laws of a perfect gas; if we call this temperature τ we easily have

$$\tau = \frac{av}{bv - R}.$$

The values of b and a are given by Profs. Ramsay and Young (*loc. cit.* p. 441); the values of τ were thence obtained, and the results are given in the following table:—

Volume.	τ.	Volume.	τ.	Volume.	τ.
300	1035	17	792·2	4	815·3
250	860·6	16	795·2	3·7	813·7
200	827·3	15	795·5	3·3	796·7
150	841·1	14	796·7	3·0	769·7
100	797·5	13	798·0	2·75	735·5
75	766·9	12	800·7	2·5	681·9
50	783·8	11	801·6	2·4	662·6
40	774·3	10	802·9	2·3	643·2
30	775·3	9	805·5	2·2	621·5
25	780·2	8	805·0	2·1	597·2
20	790·1	7	805·6	2·0	571·2
19	789·2	6	805·5		
18	789·4	5	807·2		

In calculating this table, the value of R was taken $=\frac{1}{\cdot 00119}$.

An examination of the table shows that τ remains nearly constant for all large volumes down to about vol. 3; its numerical value throughout this range is roughly 800. The only exception occurs in the case of vol. 300, but the value of τ is here so erratic that it is clearly subject to a large experimental error.

We have still to try how far the formula found for isopentane can be made by alterations of the constants to suit the experimental results with ether. For isopentane it was found that fairly good concordance with experiment could be secured by the use of the formula

$$p=\frac{\mathrm{RT}}{v}\left\{1+\frac{e}{v+k-gv^{-2}}\right\}-\frac{l}{v(v+k)},$$

where R, e, k, g, and l are constants characteristic of the gas*. I have calculated suitable numerical values for these constants in the case of ether from Ramsay and Young's original experimental data; the values are given in the following table, and those of isopentane are added for the sake of comparison:—

	Ether.	Isopentane.
R	840·34	863·56
e	7·485	7·473
k	3·188	3·636
g	4·4539	6·2318
l	5,095,070	5,420,800

The formula was tested by drawing a system of continuous isothermals giving pv plotted against $v^{-\frac{1}{2}}$, and then the experimental values were put in as dots; there is on the whole a fair agreement between calculation and experiment, as may be seen on inspection of the diagram. It is generally difficult in these investigations to know how much may be reasonably allowed for experimental errors. Fortunately in

* This formula was incorrectly stated in my former paper "On the Isothermals of Isopentane;" the correct form is that given above.

this instance we have a clue to guide us, as Messrs. Ramsay and Young in testing their linear law published tables comparing pressures found with pressures calculated (*loc. cit.* pp. 438–440 and pp. 442–445), and from these it is seen that they were willing to allow over 1 per cent. as a possible experimental error. In this connexion they remark: "It is to be noticed that the greatest divergence is at the temperatures 250° and 280°, but the deviations are in opposite directions and must therefore be ascribed to experimental error" (*loc. cit.* p. 444).

I likewise found in testing my formula that the greatest divergence is at temperatures 280°·35 C. and 250° C., and that the deviations are in opposite directions, and therefore consider it justifiable to attribute them mainly to the same cause. For the remaining temperatures discrepancies occur fairly often of over 1 per cent., but none so great as 2 per cent., so that they still seem to lie within the limits of experimental error.

Finally we may infer that both the general conclusions obtained in the former paper with regard to isopentane hold good also in the case of ether.

Discussion.

Prof. Ramsay said that experimental errors might account for some of the lack of agreement between proposed formulæ and direct observation of the behaviour of gases. Isopentane was probably a better investigated body than ether, for it was simpler. Ether tended to form complex molecular groupings, but isopentane was probably a mono-molecular liquid.

Prof. Perry did not quite agree with the author's conclusions. It was necessary to distinguish between a formula founded on a physical hypothesis and a mere empirical formula. The author had assumed that the Ramsay and Young formula was very exact; its originators did not put it forward as being infinitely exact. Probably the best test for such a formula as that under discussion would be derived from some thermo-dynamical conclusion deduced

from it. The Rose-Innes formula, with five constants and inplying discontinuity, was to be distrusted, for there was no such thing as discontinuity in the problem. In any case, an empirical formula should have a very simple form.

Mr. ROSE-INNES admitted that a formula founded on sound hypothesis was to be preferred to empirical expressions. But mathematicians had not yet provided a hypothesis applicable to a substance whose molecular arrangement was so complicated as that of ether. Mathematicians must therefore improve their methods before working formulæ could be deduced from their hypotheses. The use of an empirical formula with five constants was justified by Kepler for the planetary orbits. Kepler used that formula with no other justification than his experience that an ellipse fitted his observations better than a circle. Similar instances might be cited from recent work on the theory of solution, and osmotics.

Mr. JOHNSTONE STONEY was disposed to look for a mathematical cause for the cusp; it was improbable that the physical change was so abrupt as that represented graphically by the author. The question might be tested by plotting the two curves $y=V^{-\frac{1}{3}}$ and $y=aV^2$, and by observing whether these also suggested discontinuity.

V. *The Failure of German Silver and Platinoid Wires.* *By* ROLLO APPLEYARD *.

THE object of the present paper is to direct attention to the serious mechanical defectiveness of certain alloys used for electrical wires. The question is the more important because mechanical weakness implies electrical instability. In what follows I endeavour to bring together facts enough to indicate the general behaviour of german-silver and platinoid, and especially to settle the case as regards the conditions external to the wires.

It is necessary to remember that great differences exist between different samples of the same nominal quality of alloy; so that, of a hundred pounds' weight of wire, ninety-nine may be proof against all ordinary climates and conditions to all time, while the remaining pound may become fragile under the same treatment in a few weeks.

Many suggestions have been made to me, and are yet current, as to the cause of the failure of these wires. A well-known expert in electrical matters attributes it to sulphur in the ebonite of the electrical apparatus. It will, however, be shown that failure occurs apart from all ebonite and sulphur. Paraffin-wax has been mentioned by another authority as the probable cause of these defects. Instances will be given to prove that paraffin-wax is an absorbent of moisture, and consequently useless for purposes of protection against climate; but I have no evidence that it is aggressively harmful to the wires. Chemists have variously attributed the failure to the presence of zinc, copper, antimony, arsenic, sulphur, phosphorus, and tungsten in the wires: the chemistry of the question must therefore be regarded as *sub judice*.

Tropical heat and moisture accelerate failure, but not in all cases. Moisture has been proved by Brereton Baker to be essential to certain chemical combinations; and there is evidence enough in what follows that the same agency is effective in bringing about changes in the constitution of

* Read November 26, 1897.

alloys. The deteriorating effect of heat and moisture is not limited to mixed metals; for I have recently examined a large tube made of electrolytic copper which, under the influence of steam, quickly became very seriously "pitted." Another tube, of the same electrolytic copper, became similarly "pitted" when used for conveying sea-water. When these were replaced by tubes of ordinary copper, there was no further trouble.

During the past six years some very remarkable instances of the failure of alloys have come to my notice, particularly with regard to wires of german-silver and platinoid used in the construction of resistance-coils. Specimens of these wires, insulated with white silk, were submitted to various conditions of climate; they were sent respectively to India, Brazil, Chile, Peru, Ecuador, Nicaragua, Mexico, and Texas. For comparison and reference, similar wires were in some instances kept in England. Several thousands of bobbins were thus distributed in widely different latitudes, *i. e.*, to Valparaiso, Iquique, Chorillos, Santa Elena, Panama, San Juan del Sur, Salina Cruz, Coatzacoalcos, Vera Cruz, Galveston, Pernambuco, Bahia, Rio Janeiro, Monte Video, and Calcutta.

A few years after this distribution faulty bobbins were reported from San Juan del Sur, Santa Elena, Panama, Vera Cruz, Bahia, and Calcutta. These six towns all lie on or near sea-coasts, and they are nearly on the same terrestrial isotherm, *i. e.* the isotherm including the area of high terrestrial mean temperature, 25° C.

In all cases of failure the alloy had become brittle and the wires had broken, not only at the outer layers, but also within the coils. The following is a short history of the wires that failed:—

(1) In April 1891 some german-silver wire, 16 mils diameter, doubly covered with silk, was wound on small boxwood bobbins and then treated with paraffin-wax. Each bobbin was afterwards lapped with a strip of leather. They were all packed in tin-lined soldered cases containing straw, and were shipped to Vera Cruz, on the Mexican Coast—a voyage of twenty-six days. A month after leaving England the cases were unpacked. It was then noticed that the leather lappings of the bobbins had deteriorated, the silk coverings were discoloured, the german-silver had become brittle, and breaks

had occurred in the wire. The packing-straw showed signs of dampness. The case had been stowed in the hold with other cargo in rather a hot part of the ship. No current was ever sent through this wire except the small fraction of a milliampere used momentarily while testing the resistance before shipment.

(2) In February 1891 similar bobbins, of nominally the same german-silver wire, were shipped to Valparaiso and Iquique. These were not treated with paraffin-wax, they were merely lapped with leather. During the Chilian war the apparatus was dismantled. I have had no information as to the state of these bobbins since January 1893, when the wire was reported to be in good condition. The mean temperature of Valparaiso is lower by about 10° C. than the mean temperature of Vera Cruz.

(3) In June 1895 bobbins of platinoid were shipped to the town of Bahia, on the Brazilian coast. The wire was silk-covered and treated with paraffin-wax. No leather or ebonite was used in connexion with this apparatus. An outer box of mahogany protected the bobbins from insects. At the end of a very wet and stormy season the wire failed. It was reported "faulty" in January 1896, about six months after its arrival in Brazil. When unwound, the wire showed several fractures; it had become "short" locally. Bahia lies almost on the same isotherm as Vera Cruz, 25° C.

(4) Between the months of June and August 1893 bobbins of platinoid were shipped to Valparaiso, Iquique, and Chorillos. The wire was silk-covered and treated with paraffin-wax. Each bobbin had an outer tube of ebonite. There have been no reports of failure from any of these towns. This goes to prove that the presence of ebonite is not itself sufficient to account for the fracture of platinoid wires.

(5) In November 1893 similar bobbins were shipped to Galveston and Coatzacoalcos: no faults have, so far, developed in them. Here, again, ebonite has done no evident harm. Coatzacoalcos is nearer to the equator than is Vera Cruz.

(6) In December 1894 similar bobbins were shipped to Santa Elena, on the coast of Ecuador, and to San Juan del Sur, in Nicaragua. Two years later two of the Santa Elena bobbins were reported "faulty," and the same number failed at

San Juan del Sur during the same time. Both towns are within about 10 degrees of latitude of the equator.

(7) In September 1895 similar bobbins were shipped to Panama. The wire of one was reported "broken" in November 1896.

(8) Between October 1894 and September 1895 about a thousand bobbins of platinoid wire, made and protected as above described, treated with paraffin-wax and sheathed with ebonite, were shipped to Pernambuco, Rio Janeiro, and Salina Cruz. No reports of failure have, so far, been received from these towns. They are all within the tropics.

(9) In March 1895 a voltmeter, wound with silk-covered german-silver wire, was shipped to Calcutta. The wire was not treated with paraffin-wax, and no ebonite was used in connexion with it. In February 1897 every inch of the wire was rotten. As the coil was unwound, the wire fell to pieces. Slight electrical heating had possibly accelerated the structural change of the alloy.

(10) The specimens of various german-silver and platinoid wires on bobbins kept in England are still quite good. This applies to the wire such as was used for the voltmeter, as well as to the platinoid such as was used for the resistance-coils. This autumn I selected some of these home-specimens and formed them into small coils. They were then exposed just above the surface of a tank of water that was boiled all day and allowed to cool all night. This process was continued for six weeks with no apparent deteriorating effect upon any of the coils.

(11) A piece of bare platinoid wire kept for some years in the laboratory has become discoloured, and there are several black spots in it; but I can find no "short" places and no mechanical weakness anywhere.

(12) Some suspended helices of bare german-silver wire have broken in several places; these have had current through them from time to time. One of the fractures occurred at or very near a brazed joint. The helices were under slight torsional stress. There were no fractures in the short horizontal german-silver wires connecting these helices.

The constitution of german-silver is stated to vary as follows:—

Copper	50–66	parts.
Zinc	19–31	„
Nickel	13–18	„

Platinoid is generally described as german-silver with two per cent. of tungsten. According to Brannt, the tungsten, in the form of phosphor-tungsten, is first melted with a certain quantity of copper. The nickel is next added, then the zinc, and finally the remainder of the copper. In order to remove phosphorus and a portion of the tungsten, both of which separate as dross, the resulting compound is several times remelted. It is probable that traces of arsenic and phosphorus are present in the alloy.

If nickel is in too great proportion in nickel-copper alloys, oxygen is absorbed during fusion, and liberated on cooling. The result is a porous metal. Again, if the temperature of casting is too high, or if the cooling is irregular, cavities may be expected in the final alloy.

The surprising diminution in tensile strength produced by traces of impurities, for some alloys, has been studied by Prof. Roberts-Austen and other metallurgists. A very small percentage of arsenic in nickel-copper-zinc compounds causes extreme brittleness. So also does a small addition of lead or iron. And as arsenical nickel ore is the source of much of the nickel of commerce, the failure of german-silver and platinoid may possibly be due to traces of arsenic.

At this point, however, I propose to discriminate between two kinds of brittleness, my object being to simplify the discussion of the problem. That such a distinction is necessary will perhaps be best illustrated by an example. Prof. Roberts-Austen demonstrated that the addition of 1 per cent. of lead reduces the tensile strength of gold by more than two-thirds. Similarly, he observed that arsenic renders gold very fragile, and that 0·2 per cent. of bismuth *instantly* converts gold into an alloy that crumbles under the die. In all these cases, brittleness is characteristic of the alloy from the moment of solidification; it may, in fact, be regarded as a definite function of the atomic volumes of the constituent elements. I propose to call this "primary" brittleness.

But the brittleness of german-silver and platinoid is of a

different order. It is a *subsequent* phenomenon. For example: the alloys from which electrical wires are made are necessarily strong in the first instance; if they were mechanically weak, *i. e.*, if they possessed "primary" brittleness, they would fail in the process of "drawing" through the die. Hence, the distinction between "primary" and "secondary" brittleness. The one is an accident of birth, the other is a disease that develops with age and circumstance.

The fracture of platinoid, as seen under the microscope, is granular or crystalline. In new and good specimens the colour is silver-grey, and uniform except for a few well-defined patches of brown. Occasional specimens of brittle wire retain, in parts, the silver-grey fracture. As a rule, however, bad specimens have, encroaching on the silver-grey area, patches of purple, yellow, copper-colour, and brown, with frequent cracks and fissures that communicate more or less with the surface of the wire. Very bad specimens have, at times, scarcely any of the silver-grey colour at the fracture. The patches of dark-purple, yellow, and copper-colour are clearly defined, as though each of them represented a distinct alloy, or a separate mass of "liquated" metal. On a particular bobbin, brittleness is often restricted to a foot or two of the wire. Similar remarks apply to the appearance of the fractures of german-silver.

The crevices and fissures in bad specimens are easy to observe with a microscope. They are probably developed during wire-drawing, as the result of pores in the cast metal. Another explanation of them may be looked for in the "liquation" of some of the metals constituting the alloy. By this action, the portion of the alloy that first solidifies rejects the yet molten portions, as ice rejects foreign matter. As cooling continues, the various constituents become isolated, homogeneity is lost, and I think we may fairly assume that, in consequence, the strength of the material varies from point to point of its mass, so that in passing afterwards through the die, the weaker constituents give way, and the general structure is loosened. Crystallization and internal electrical actions may also result in local weakness, crevices, and fissures.

Keeping in mind the existence of these fissures it is a

simple matter to account for the fact that wires deteriorate more quickly under a small stress than when they are submitted to no stress. Apart from the mechanical weakening, it is evident that any extending force opens the crevices and makes way for air and moisture—it may also facilitate crystallization. And clearly, if the wires are being used for electric currents, as is the case with resistance-coils on arc-light circuits, great local heating occurs at these weakened sections, and fracture is inevitable. This was probably the fate of the voltmeter-wire mentioned in example (9) above. It was almost certainly the fate of the suspended helices (12).

From what precedes, there is plenty of evidence that secondary brittleness is fostered by tropical heat and moisture. A good wire, free from incipient cracks, may last for years, or may be proof against heat and moisture, as were the wires in example (10). But if there are any fissures, there is no doubt that moisture intrudes upon the alloy through these capillary channels, with destructive effect.

There is a kind of tradition that paraffin-wax is a perfect safeguard against moisture; as a matter of fact it is highly absorbent. The wax on the bobbins returned from the tropics is completely choked with moisture. Shellac, or the old-fashioned sealing-wax varnish, resists much better. There is no evidence to show that ebonite is harmful to german-silver or platinoid, but it may be well to keep the metal out of actual contact with ebonite.

Can metallurgists tell us the difference, in constitution and structure, between a german-silver wire that decays in four weeks, and another that under similar conditions never fails? Or, what is even more important, can they make us platinoid that shall never fail? If they cannot, it becomes necessary to surrender those cheaper and better electrical materials, and fall back upon the more expensive alloys, beginning at platinum-silver. If sufficient time and means were at the disposal of metallurgists they might discover the secret of permanence in alloys. Germany, with the advantages of a National Laboratory, has already attacked the question, and "manganin" is the result. Its adoption there as a satisfactory alloy is directly due to work done upon it by the Reichsanstalt. But it has yet to be proved that manganin

will endure the conditions imposed by the tropics. Moreover, the doubt arises as to whether it is desirable to obtain from abroad material that ought to be produced by our own country. British cable-manufacturers are already importing thousands of tons annually of sheathing-wire from Germany; and it seems probable that, for the want of a National Laboratory, instrument-makers will now get their resistance-wire from that same adventurous foreign source.

Discussion.

Prof. Ayrton said the paper had raised the extremely interesting question of the permanence of metals used for resistance-coils. Some time ago he had immersed bare platinoid wires in running water in metal tanks, and the wires all broke in short pieces. He thought, at the time, this might be due to electrolysis. On another occasion he had found that, by raising the temperature of platinoid to a dull-red heat in the air, by an electric current, any acquired faults in the wire were corrected, and the original resistance and flexibility were restored. Even when such metals are in good condition, the resistance-temperature curve does not return upon itself; it encloses a loop, indicating two distinct values for resistance at each temperature. He had been told by Dr. Muirhead that coils intended for hot climates should be enclosed in air-tight metal cases. English manufacturers were still dubious in regard to manganin. In 1892 he had twenty coils made of this material, each of 1000 ohms; the wire was silk-covered. There were 2000 volts between the terminals. Their resistance had certainly not changed by 1 in 1000, although there was some amount of vagueness regarding the fifth figure, which might be due to molecular alteration, for they were heated more than was good for resistance-coils. He confessed that this manganin had come from Germany.

Dr. S. P. Thompson mentioned an alloy that was proposed in Germany under the name of "Constantine." He would like to know whether any information could be obtained as to the employment of cast-iron wire. It was a metal that in some respects commended itself. He had observed the

failure of some german-silver coils, but he had generally attributed it to rough handling.

Mr. W. WATSON referred to the recent work done at the Reichsanstalt with regard to german-silver and platinoid. It was there found that all alloys containing zinc were liable to erratic changes of resistance, and were unsuitable for standard coils. Moreover, even the slight amount of zinc introduced into manganin during soldering with soft solder robbed that alloy of its constancy. Silver solder, containing 75 per cent. of silver, should be employed for manganin. If Prof. Ayrton's coils were soldered with soft solder, that was sufficient to account for the change in the fifth figure. Shellac varnish was undoubtedly the best protection for coils. Absolute alcohol should be used as the solvent, and the coils should afterwards be heated for some hours at 140° C. If a heating-current was passed through german-silver or platinoid coils immersed in water, the general result was to produce brittleness. Mr. Watson then described a thermostat which he had contrived for "ageing" the coils after applying the shellac varnish. A hot-air oven contains a thermometer with a platinum contact at the 140° mark and an 8 c.p. lamp. The thermometer is in circuit with a relay actuating a mercury-key for the 8 c.p. lamp. The key consists of two mercury-cups, and a corresponding U-piece of copper, inverted, one limb to each cup. It is important to keep the heating-circuit always made; for this purpose a 32 c.p. lamp is permanently connected between the two cups.

VI. *On Lord Kelvin's Absolute Method of Graduating a Thermometer.* By J. ROSE-INNES, *B.Sc.**

IN a paper "On the Thermal Effects of Fluids in Motion" Lord Kelvin has given the cooling effects exhibited by various gases in passing through a porous plug; and he found that the effects for any one gas kept at the same initial temperature were proportional to the difference of pressure on the two sides of the plug (Reprinted Papers, vol. i. pp. 333–455). He also found that the cooling effect for any one gas per unit difference of pressure varies as the inverse square of the absolute temperature; and this rule succeeds very well in the case of air. For carbonic acid, however, the results furnished by this rule are not so satisfactory, as may be seen by inspecting the table given by him comparing the actual with the theoretical cooling effect (*loc. cit.* p. 429). Moreover in the case of hydrogen it is found that there is a *heating* effect, which increases, if anything, when the temperature rises, so that here the law of the inverse square of the temperature is wholly inapplicable. It seemed to me that it might be possible to hit upon some simple algebraic expression which should reproduce the experimental results rather better than Lord Kelvin's rule does; and in fact it was found that a satisfactory agreement between observation and calculation might be obtained by putting

$$\text{cooling effect} = \frac{\alpha}{T} - \beta,$$

where α and β are constants characteristic of the gas, and T is the absolute temperature.

The following values of α and β were found from the experimental data:—

	α.	β.
Air	441·5	·697
Carbonic acid . .	2615	4·98
Hydrogen . . .	64·1	·331

A comparison of the actual results with those calculated by the new formula is given in the following table:—

* Read December 10, 1897.

Name of Gas.	Temp.	Actual cooling effect.	Calculated cooling effect (Kelvin's formula).	Calculated cooling effect (New formula).
Air	0°	0°·92	°·92	·920
	7·1	·88	·87	·879
	39·5	·75	·70	·716
	92·8	·51	·51	·510
Carbonic Acid......	0	4·64	4·64	4·60
	7·4	4·37	4·40	4·35
	35·6	3·41	3·63	3·49
	54·0	2·95	3·23	3·02
	93·5	2·16	2·57	2·16
	97·5	2·14	2·52	2·08
Hydrogen .	4·5	−0·100	......	−·100
	91·0	− ·155	......	−·155

It will be seen that the formula proposed in this paper, regarded simply as an empirical formula, is more efficient than Lord Kelvin's; there is nothing astonishing in this as it contains two disposable constants instead of only one. But it has the following further advantages:—

(i.) It includes the three cases of air, hydrogen, and carbonic acid under one form, and therefore enables us to treat them all in one common investigation.

(ii.) It renders more manageable the differential equation concerned in the thermodynamic scale of temperature, and leads to simpler algebraic results after integration.

This last proposition we must now proceed to prove. It is shown by Lord Kelvin that when a gas passes through a porous plug we must have

$$t\frac{dv}{dt}-v=\frac{\mathrm{JK}}{\Pi}\theta,$$

where t and v denote the temperature and volume of the gas respectively, K its specific heat, θ the cooling effect per atmo of differential pressure, Π the value of one atmo, and J the value of Joule's mechanical equivalent (Reprinted Papers, vol. iii. p. 179). Hence

$$\frac{1}{t}\frac{dv}{dt}-\frac{v}{t^2}=\frac{\mathrm{JK}}{\Pi}\frac{\theta}{t^2}$$

$$\frac{d}{dt}\left(\frac{v}{t}\right)=\frac{\mathrm{JK}}{\Pi}\left\{\frac{\alpha}{t^3}-\frac{\beta}{t^2}\right\}.$$

Estimate of the Absolute Value of the Freezing-point of Water.

Integrate the last equation between the limits t_0 and t_1, and we obtain

$$\frac{v_1}{t_1}-\frac{v_0}{t_0}=\frac{\mathrm{JK}}{\Pi}\left\{\frac{\alpha}{2t_0^2}-\frac{\alpha}{2t_1^2}-\left(\frac{\beta}{t_0}-\frac{\beta}{t_1}\right)\right\}$$

$$=\frac{\mathrm{JK}}{\Pi}\left\{\frac{1}{t_0}-\frac{1}{t_1}\right\}\left\{\frac{\alpha}{2}\left(\frac{1}{t_0}+\frac{1}{t_1}\right)-\beta\right\}$$

$$=\frac{\mathrm{JK}}{\Pi}\left\{\frac{1}{t_0}-\frac{1}{t_1}\right\}\frac{\theta_1+\theta_0}{2},$$

where θ_1 and θ_0 are the values of θ at the temperatures t_1 and t_0 respectively.

Multiply the equation by $t_0\,t_1$ and we get

$$v_1 t_0-v_0 t_1=\frac{\mathrm{JK}}{\Pi}(t_1-t_0)\frac{\theta_1+\theta_0}{2}$$

$$(v_1-v_0)t_0-v_0(t_1-t_0)=\frac{\mathrm{JK}}{\Pi}(t_1-t_0)\frac{\theta_1+\theta_0}{2}.$$

Hence

$$t_0=\frac{t_1-t_0}{v_1-v_0}\left(v_0+\frac{\mathrm{JK}}{\Pi}\frac{\theta_1+\theta_0}{2}\right)$$

$$=(t_1-t_0)\frac{v_0}{v_1-v_0}\left(1+\frac{\mathrm{JK}}{\Pi v_0}\,\frac{\theta_1+\theta_0}{2}\right).$$

If t_1 and t_0 are taken as the boiling-point and freezing-point of water respectively, then this equation gives us the value of the freezing-point t_0 in terms of the interval t_1-t_0; it is usual, as pointed out by Lord Kelvin, to take the interval t_1-t_0 as containing 100 degrees (*loc. cit.* p. 175). It is evident that we should have obtained the same value for t_0 if θ had been constant throughout the range of temperature t_0 to t_1, and equal to $\frac{\theta_1+\theta_0}{2}$. This shows that the proper mean cooling-effect is simply the arithmetic mean of th cooling effects at the boiling-point and freezing-point. The following table gives us the value of the freezing-point derived from experiments on the three gases.

	Uncorrected estimate of temperature of freezing-point.	Correction.	Corrected estimate.
Hydrogen . . .	273·13	—·13	273·00
Air	272·44	·72	273·16
Carbonic acid . .	269·5	4·35	273·85

The first column of figures is taken from Lord Kelvin's paper (*loc. cit.* p. 177).

Thermodynamic Correction for a Constant-pressure Gas Thermometer.

Suppose now we have a temperature t lying above both t_1 and t_0, and fixed by some definite physical phenomenon. We require to know exactly how it lies with respect to t_0 and t_1.

We start as before with the differential equation

$$\frac{d}{dt}\left(\frac{v}{t}\right)=\frac{\mathrm{JK}}{\Pi}\left\{\frac{\alpha}{t^3}-\frac{\beta}{t^2}\right\}.$$

We may put this

$$\frac{d}{dt}\left(\frac{v}{t}\right)=\frac{\mathrm{A}}{t^3}-\frac{\mathrm{B}}{t^2},$$

if $\mathrm{A}=\frac{\mathrm{JK}\alpha}{\Pi}$ and $\mathrm{B}=\frac{\mathrm{JK}\beta}{\Pi}$

Integrate between the limits t_0 and t,

$$\frac{v}{t}-\frac{v_0}{t_0}=\frac{1}{2}\left(\frac{\mathrm{A}}{t_0^2}-\frac{\mathrm{A}}{t^2}\right)-\mathrm{B}\left(\frac{1}{t_0}-\frac{1}{t}\right)$$

$$=\left(\frac{1}{t_0}-\frac{1}{t}\right)\left\{\frac{\mathrm{A}}{2}\left(\frac{1}{t_0}+\frac{1}{t}\right)-\mathrm{B}\right\}.$$

Hence

$$\frac{v}{t}-\frac{v_0}{t_0}=\frac{-t_0}{tt_0}\left\{\frac{\mathrm{A}}{2}\left(\frac{1}{t_0}+\frac{1}{t}\right)-\mathrm{B}\right\}.$$

Multiply by $t_0(t-t_1)$ and we shall have

$$v\left(t_0-\frac{t_0t_1}{t}\right)-v_0(t-t_1)=\frac{(t-t_1)(t-t_0)}{t}\left\{\frac{\mathrm{A}}{2}\left(\frac{1}{t_0}+\frac{1}{t}\right)-\mathrm{B}\right\}.$$

Similarly by interchanging t_0 and v_0 with t_1 and v_1, we might obtain

$$v\left(t_1 - \frac{t_0 t_1}{t}\right) - v_1(t-t_0) = \frac{(t-t_1)(t-t_0)}{t}\left\{\frac{A}{2}\left(\frac{1}{t_1}+\frac{1}{t}\right) - B\right\}.$$

Subtract the equation last but one from the last,

$$v(t_1-t_0)-(v_1-v_0)t+v_1t_0-v_0t_1 = \frac{(t-t_1)(t-t_0)}{t}\,\frac{A}{2}\left(\frac{1}{t_1}-\frac{1}{t_0}\right).$$

We may write this

$$(v-v_0)(t_1-t_0)-(v_1-v_0)(t-t_0) = \frac{(t-t_1)(t-t_0)}{t}\,\frac{A}{2}\left(\frac{1}{t_1}-\frac{1}{t_0}\right).$$

Divide by v_1-v_0 and transpose,

$$t-t_0 = \frac{v-v_0}{v_1-v_0}(t_1-t_0) + \frac{(t-t_1)(t-t_0)}{t}\,\frac{A}{2(v_1-v_0)}\left\{\frac{1}{t_0}-\frac{1}{t_1}\right\}.$$

Finally,

$$t=t_0 + \frac{v-v_0}{v_1-v_0}(t_1-t_0) + \frac{JK(\theta_0-\theta_1)}{2\Pi(v_1-v_0)}\,\frac{(t-t_1)(t-t_0)}{t}.$$

This last expression for t may be said to consist of three parts :—

(i.) There is the quantity t_0, whose absolute value may be considered to have been determined once for all by means of its own equation. We may regard it as known numerically with sufficient accuracy.

(ii.) There is the term $\frac{v-v_0}{v_1-v_0}(t_1-t_0)$. This term gives us the degrees above freezing-point on the equi-expansion method of graduation. It is what is usually called the "temperature."

(iii.) The quantity $\frac{JK(\theta_0-\theta_1)}{2\Pi(v_1-v_0)}\,\frac{(t-t_1)(t-t_0)}{t}$ is due entirely to the Joule-Thomson effect. It may be regarded as a correction necessary owing to the deviation of the substance from a perfect gas. It is this quantity which is calculated and tabulated so as to give us the means of arriving at the absolute scale. We notice that in order to calculate this term, we require to know the value of t, the very quantity we are seeking to find. Sufficiently accurate, however, for the purpose of calculating this small term, will be the value of t found by means of a first approximation.

Thermodynamic Correction for Constant-volume Gas-Thermometer.

We will now compare the indications of a gas-thermometer kept at constant volume with the thermodynamic scale of temperature.

Let us start as before with the differential equation

$$\frac{d}{dt}\left(\frac{v}{t}\right)=\frac{A}{t^3}-\frac{B}{t^2}.$$

Integrate this between the limits t and ∞ and we have

$$P-\frac{v}{t}=\frac{1}{2}\frac{A}{t^2}-\frac{B}{t},$$

where P is the value of $\frac{v}{t}$, as v and t are made to grow indefinitely large with p constant. To determine the form of P we must appeal to experiment. We know that when a gas is made to expand to larger and larger volumes it obeys Boyle's law more and more closely; hence we infer that when v and t are made indefinitely large, the value of $\frac{v}{t}$ must vary inversely as the pressure. We may therefore write

$$P=\frac{C}{p}$$

where C is a constant characteristic of the gas.

Writing in this value we have as the complete solution

$$\frac{C}{p}-\frac{v}{t}=\frac{1}{2}\frac{A}{t^2}-\frac{B}{t}.$$

Multiply this by pt, and transpose,

$$pv=Ct-p\left(\frac{1}{2}\frac{A}{t}-B\right).$$

If we neglect the Joule-Thomson effect we have as a first approximation $pv=Ct$, or $t=\frac{pv}{C}$, which is the value for t in the case of a perfect gas; and this approximate value for t may be used in the term involving the Joule-Thomson effect on the right-hand side of the equation. We then get

$$pv=Ct-\frac{CA}{2v}+pB.$$

This equation represents our *second* approximation to the correct formula. We may also write it

$$p(v-\mathrm{B}) = \mathrm{C}t - \frac{\mathrm{CA}}{2v}.$$

Imagine now that we keep v constant, and use the equation to determine t when p is measured. Let the suffix 0 refer to the freezing-point, and the suffix 1 to the boiling-point as before. We then have

$$p_0(v-\mathrm{B}) = \mathrm{C}t_0 - \frac{\mathrm{CA}}{2v},$$

$$p_1(v-\mathrm{B}) = \mathrm{C}t_1 - \frac{\mathrm{CA}}{2v}.$$

By subtraction

$$(p_1-p_0)(v-\mathrm{B}) = \mathrm{C}(t_1-t_0),$$

$$(p-p_0)(v-\mathrm{B}) = \mathrm{C}(t-t_0),$$

$$\therefore \quad \frac{p-p_0}{p_1-p_0} = \frac{t-t_0}{t_1-t_0};$$

or

$$t = t_0 + \frac{p-p_0}{p_1-p_0}(t_1-t_0).$$

To the degree of approximation to which we are working, therefore, there is no thermodynamic correction needed for a constant-volume gas thermometer. There may be a correction involving *squares* of small quantities, which would appear on a nearer approximation. Such a correction, however, would not be worth taking into account in the case of a thermometer constructed with air or hydrogen, as the unavoidable errors of experiment would certainly be much larger than the correction. It is satisfactory to know that for all practical purposes absolute temperature is to be obtained with very great accuracy from Regnault's thermometric system by simply adding the value of t_0 to his numbers for temperature on the centigrade scale.

This result differs from that obtained by Rowland *, who employed Lord Kelvin's law of the inverse square of the temperature, and inferred that there was a correction needed

* Proceedings Amer. Acad. Arts & Sciences. xv. (n. s. vii.) p. 114.

involving the first power of the cooling-effect. Both his investigation and my own involve the assumption that an empirical formula found to hold through a short range of temperature can be used for any temperature however high, and hence neither his result nor mine is conclusively established; but it seems interesting to show that the employment of a new expression, at least as good as Lord Kelvin's, for the cooling-effect, leads to a smaller value for the correction.

Discussion.

Dr. S. P. Thompson said the empirical expression,

$$\left(\frac{\alpha}{T}-\beta\right),$$

indicated that at some particular temperature the cooling effect vanished; that was a point suggestive of useful results if investigated by experiment.

Mr. J. Walker read a communication from Mr. Baynes on the paper, and remarked on the desirability of adopting two constants. He thought that further experiments should be made to discover how specific heat at a constant temperature depends on temperature. The calculated values for hydrogen were too few to be taken as evidence of the validity of the rule.

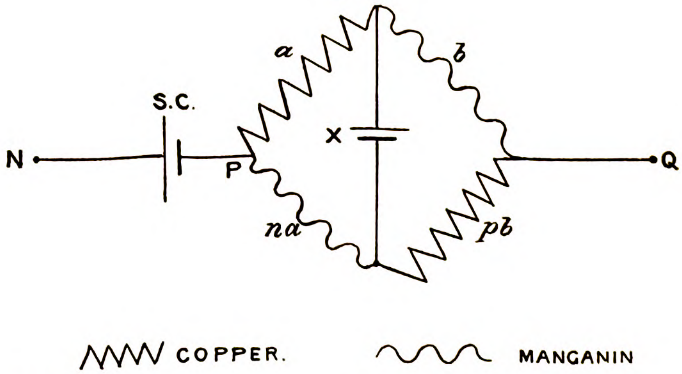
Mr. Rose-Innes, in reply, said that from what was known of hydrogen, it might be expected to behave at ordinary temperatures as air behaves at higher temperatures. His object was, if possible, to include in one formula the case of the three investigated gases. This was much better than having a separate formula for each gas. Whether or not hydrogen was confirmatory with air and carbonic acid might be considered as *sub judice*; it required further experimental data to test the formula in that case.

VII. *On Apparatus for Self-acting Temperature Compensation of Standard Cells.* By ALBERT CAMPBELL, *B.A.**

In ordinary testing work when using a Clark cell as the working standard of difference of potential, it is desirable to get rid of the trouble of continually having to make the temperature correction. Some time ago ('The Electrician,' pp. 601–603, September 6, 1895) I proposed and investigated three methods by which a self-acting correction could be attained. Quite recently I have constructed the apparatus for two of the methods, and I have found it an easy matter to reach the accuracy aimed at (viz. maximum error 1 in 2000).

The arrangement of the first piece of apparatus is shown in fig. 1.

Fig. 1.



S.C. is the standard cell. Four resistances of values a, b, na, and pb are connected as in a Wheatstone's bridge with an auxiliary cell X, whose voltage must be known within 5 or 6 per cent. A Leclanché cell is found suitable for this. The resistances a and pb are of a metal such as copper or iron with a large temperature-coefficient, while b and na are of manganin or a similar alloy. As shown in the paper cited above, we can choose n and p so as to have a constant voltage of, say, 1·400 volts between N and Q at all temperatures from 0° to 20° C. when S. C. is a Clark cell. The arrangement thus not only corrects for temperature but can be made to

* Read December 10, 1897.

give a more even figure by cutting off the 0·034 volts at the same time. In one of the coils exhibited the resistances have the following values at 15° C :—

$$a = 48{\cdot}46 \text{ ohms.}$$
$$na = 482{\cdot}7 \quad ,,$$
$$b = 63{\cdot}63 \quad ,,$$
$$pb = 495{\cdot}0 \quad ,,$$

As the absolute values of a and b are of no importance so long as they are high enough to keep the auxiliary Leclanché cell from running down, the copper coils a and pb did not require to be adjusted but were merely measured; their resistances were reduced to 15° C. and the manganin coils na and b were then adjusted to give $n=9{\cdot}96$, $p=7{\cdot}78$*. It is clearly a great saving of trouble not to have to adjust the coils which have a high temperature-coefficient. The four coils were wound on a brass cylinder 7 or 8 centim. in diameter, and then well protected with silk ribbon, four leads being brought out. To test the whole directly the cylinder well, wrapped in wadding, was placed in a tin canister, the lid of which was then *soldered on* so as to be watertight. The leads were brought out through a long tube soldered to the lid. A thermometer also passed down this tube into the canister. A Leclanché cell was connected as in fig. 1. The tin was then well immersed in water at various temperatures, and steady temperatures inside the tin were got by stirring the water until a thermometer in it showed the same temperature as the one inside the tin. The potential-difference from P to Q was measured by a potentiometer method at various temperatures with the following results :—

Temperature.	P.-D., volt.
15°·5 C.	0·0338
34·2 C.	0·01472
48·5 C.	0·0000

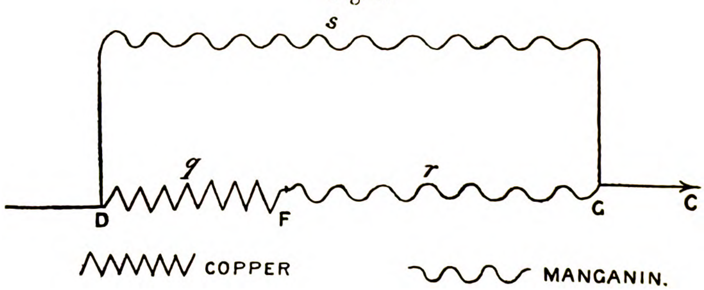
* The numerical values of p and n given in my paper mentioned above are not quite right owing to a slip in signs in the equation there given.

It will be seen that the actual observed values of the auxiliary voltage are just what would be required along with the Clark cell to give 1·400 volts at all ordinary temperatures.

The arrangement in fig. 1 was chosen so as to give the maximum possibility of correction. Mr. C. W. S. Crawley has pointed out to me that some trouble is saved in the construction of the coils by making only one of the four of copper. This also simplifies the calculation of the ratios n and p, and still gives sufficient variation to balance that of a Clark cell. It is convenient to place the cell inside the wide brass tube on which the coils are wound.

In the other arrangement exhibited three resistance-coils are connected as in fig. 2.

Fig. 2.



The resistances r and s are of manganin, and q is of copper. A current C is sent from D to G from an outside source. The values of q, r, and s are so chosen that a fixed value of C produces *at all temperatures* a potential-difference from F to G which would balance the voltage of the standard.

In the specimen shown $3{\cdot}75\ q=r+s$ and $r=s$. The actual values at 15° C. are

$$r=s=3{\cdot}6335 \text{ international ohms.}$$
$$q=1{\cdot}9382 \quad \text{,,} \quad \text{,,}$$

A current of 1·000 ampere gives a balance with a Clark cell at F and G at all ordinary temperatures (to within 1 in 2000). This definite current, obtainable at all temperatures, can be passed through known resistances and so used to measure unknown voltages by potentiometer methods.

If the resistance r be subdivided other exact currents such as 2, 3, 4 ... amperes can be got directly.

It should be noted that although I have made special reference to the Clark cell, either of the two forms of compensator may be arranged for any other type of standard cell so as to wipe out the greater part of the temperature variation. For instance, if a Carhart 1 volt cell has a coefficient of 0·0097 per cent. per degree C., used with a compensator it would not vary by more than 0·001 per cent. per degree. It will thus be seen that in some cases a compensating arrangement such as I have described above would prove of value not only in ordinary testing work but also in measurements where much higher accuracy is required.

Discussion.

Mr. Swinburne said that twelve or thirteen years ago he had given a good deal of thought to compensation by wires of different temperature-coefficients. The first thing he tried was a Wheatstone's bridge. This was compensated by making the bridge-arms of wires whose temperature-coefficients differed—as, for instance, platinoid and copper. He then applied the same principle to the compensation of standard cells, using a potentiometer method that gave direct readings, and to the compensation of voltmeters and Wattmeters. These results were published between 1885 and 1890, in the electrical journals. He believed that Mr. Evershed had also developed this idea, by putting "back" turn on voltmeters, and by other differential devices. The details of Mr. Campbell's apparatus had a few points of special interest. The way in which he connected up the bridge (3) seemed particularly worthy of notice.

Prof. Ayrton asked whether thermo-electric effects produced difficulty in the compounded arrangement.

Mr. Campbell said the system was symmetrical, and the thermal currents were consequently neutralized.

Mr. Campbell, in reply, said that he was surprised to hear Mr. Swinburne justify the retention of the uncompensated Clark cell because it is always used with a potentiometer.

He ought to have said "with a potentiometer *and a thermometer.*" This means that every time the potentiometer current is re-set a reading of the temperature has to be taken and the slider adjusted to correspond to this. This double operation, which must necessarily increase the probable error, is entirely done away with if the cell is compensated. If any part of a measurement can be rendered automatic, surely we may expect to gain both in accuracy and in quickness of work. With reference to the second apparatus, it is *not,* in itself, a compensated potentiometer, and so does not seem to be the same as that Mr. Swinburne invented some years ago. It is simply an arrangement by which a current can be set to an exact value (say 0·1 ampere) with the help of a standard cell. This current can of course be sent through resistances of invariable material, and so potentiometer measurements can be made.

VIII. *Variations in the Electromotive Force of the* H-*form of Clark Cells with Temperature.* *By* F. S. SPIERS, *B.Sc.*, F. TWYMAN, *and* W. L. WATERS.*

THE work described in this paper was intended as a completion of the investigation carried out some years ago by Prof. Ayrton and Mr. W. R. Cooper † on the effect of variations of temperature on the E.M.F. of the Board of Trade form of Clark cell. Their results showed that it was not possible, by applying the ordinary temperature-correction, to obtain the true value of the E.M.F. of the cell with a greater accuracy than 0·1 per cent. It was thought that the H-form of cell would show some improvement over the ordinary type in this respect; and this was confirmed by some results published recently by Dr. Kahle ‡. The experiments to be described in this paper were undertaken with the object of seeing how far this was correct, and in general to investigate

* Read November 12, 1897.

† Proc. Roy. Soc. December 1895.

‡ *Zeitschrift für Instrumentenkunde,* August 1893.

the behaviour of this form of cell under variations of temperature. The experiments were carried out at the Central Technical College.

The tests were made on four cells set up in accordance with the specification issued by Dr. Kahle from the Reichsanstalt*. The cells were in every way alike †, and were mounted together inside a glass vessel in paraffin oil, and during the experiments they each underwent precisely the same treatment. The diameter of each of the two legs of the cell was about ¾ inch, and their length about 1 inch.

The method used for the determination of the E.M.F. was as follows. Current from a storage-cell traversed a potentiometer-wire in series with a resistance, the value of which was such that the fall of potential along the wire amounted to one ten-thousandth of a volt per centimetre-length. The cell to be tested was joined up in opposition to the standard, and the difference in their E.M.Fs. was balanced against the fall of potential along a certain length of the stretched wire.

The standard used in comparison consisted of one or other of two cells of the Muirhead type, which had been found to have an E.M.F. approximately constant during several years. The two cells were in one case, and during the experiments were placed in a thermostat in which the temperature could be kept constant to $0^{\circ}{\cdot}01$ C. The E.M.F. of the standard was determined absolutely by balancing it against the P.D. between the terminals of a resistance of known value of about three ohms, due to a current of about half an ampere passing through it. The resistance had been made specially for the purpose of manganin wire, wound upon an ebonite frame and immersed in paraffin oil. It had been previously aged by heating, and had assumed a steady value, as shown by comparison with standard manganin coils. The current was measured by a Kelvin centiampere balance, the constant of which had been frequently checked by means of a silver voltameter. By this means the E.M.F. could be determined absolutely, correct to 0·0001 volt.

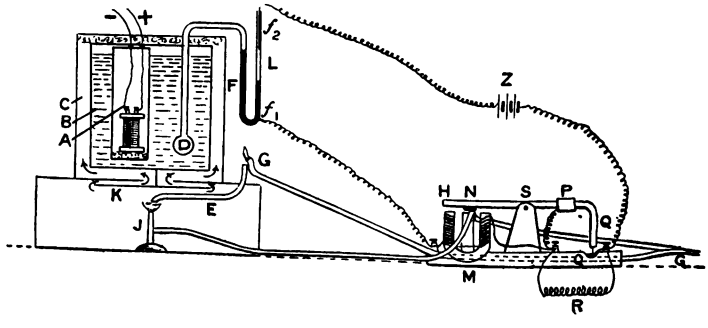
* 'Electrician,' vol. xxxi. p. 265, July 1893.

† They were set up by Mr. Cooper, Nos. 1, 2, and 4 on 19th March, 1895, and No. 3 on 12th February, 1896.

The thermostat consisted of an inner air-space, in which was placed the standard, surrounded by an outer water-bath which was heated by a gas-jet, the gas being automatically cut off when the temperature rose to a given value, and being allowed to come on again as soon as the temperature had fallen below that value.

In fig. 1, A B C are zinc vessels : A is the air-bath, containing the standard resting on cottonwool ; B the surrounding water-bath, closed at the top and containing the thermometer D ; C the enclosing jacket, between which and B the hot air from the gas-jet passes, heating the water-jacket on its way. D is an alcohol thermometer with a mercury-index F, which, on the expansion of the alcohol, is pushed up the narrow tube L, and electrically connects the two platinum wires f_1 and f_2. J is the gas-jet for heating the water-bath ; K a baffle-plate over the gas-jet, to prevent the flame heating the water-bath directly. E is an escape-tube, which serves to carry off some of the heat of the jet ; G is a little by-pass,

Fig. 1.



burning at the mouth of the tube E, and serves to relight the main jet when extinguished. M is an electromagnet ; H, its armature, is pivoted at S and weighted at P, and carries a platinum wire Q which dips into the mercury cup Q′.

When the temperature of the water-bath has reached a certain arbitrary value, the mercury-index electrically connects f_1 and f_2, completing the circuit of the electromagnet M and the battery Z. The electromagnet being thus excited, attracts its armature H, which compresses the indiarubber

tube T that feeds the main gas-jet J, between the brass piece N and the adjustable wooden block N′, and so extinguishes the light. The water-bath now cools slowly till its temperature has fallen below the fixed value, and the contact is broken at f_2 by the falling of the mercury-index. When this happens, the armature of the electromagnet is released and pulled up at once by the weight P, the gas coming on again at the jet J ; some of the gas passing up the tube E is ignited at the top by the by-pass G, and then striking back, lights the main jet J, restarting the heating, and the cycle is repeated.

As it requires a less current to hold the armature down than it takes originally to attract it, the current circuit of the magnet is so arranged that when the armature is attracted it breaks the direct-current circuit at the mercury cup Q′, and inserts into the circuit a resistance of 50 ohms, thus economizing the energy.

The temperature at which the cut-off acts can be adjusted several degrees by altering the distance which the wire f_2 projects down the tube L.

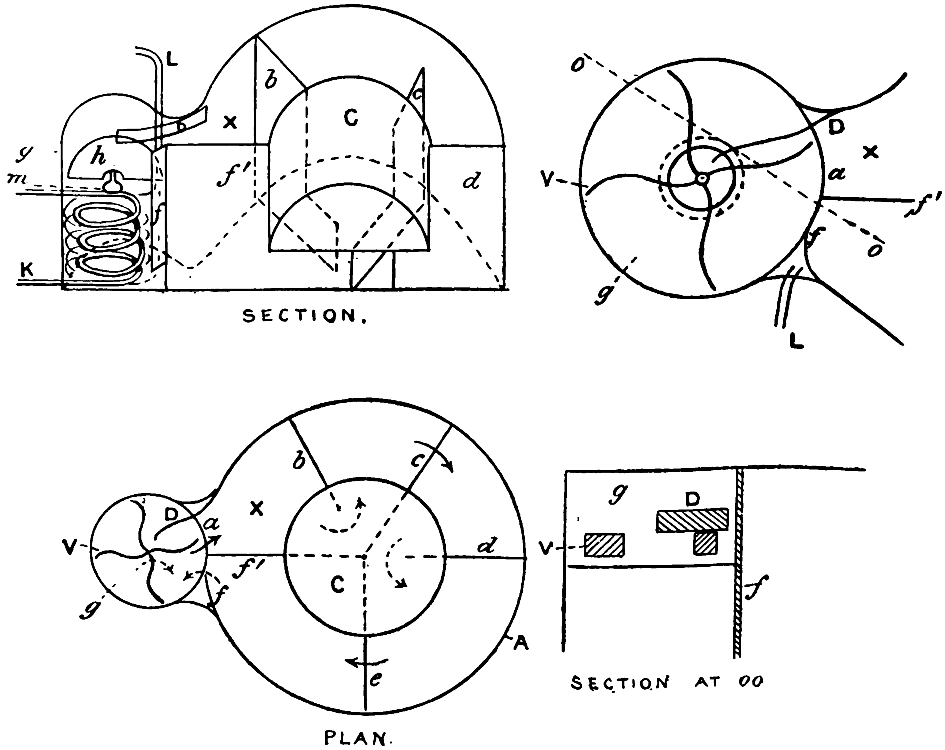
By carefully adjusting the size of the jet, the by-pass, and the height of the block N′, we could keep the temperature of the air-bath constant to 0°·01 C., the gas being cut off and coming on again about every five minutes.

To investigate the effect of temperature on the cells, it was necessary to be able to vary their temperature so as to get a uniform rise or fall. To effect this, some special form of heating-bath was needed. In its final form, this heating-bath consisted of an outer zinc bath, in which the vessel containing the Clark cells was placed, which was divided up by means of a number of partitions, reaching from the bottom of the vessel to above the surface of the oil. The oil was forced alternately under one partition and over the next, round the whole of the bath, and from the last compartment was drawn off into a separate chamber containing a heating-coil and a coil of lead tubing for an iced-water circulation, and from this chamber it was again circulates round the compartments by means of a set of centrifugal vanes.

A (fig. 2) is the outer zinc bath, containing the oil, and divide

up into compartments by means of the zinc plates, *a*, *b*, *c*, . . . C, the inner glass vessel, contains the cells, and rests about one and a half inches from the bottom of the outer bath on the parts of the partitions *a*, *b*, *c*, . . . which reach to the centre. *g* is the separate compartment containing the heating-coil L, the coil of lead tubing K, and the centrifugal pumping-vanes V. The vanes V are directly coupled to a motor with a

Fig 2.



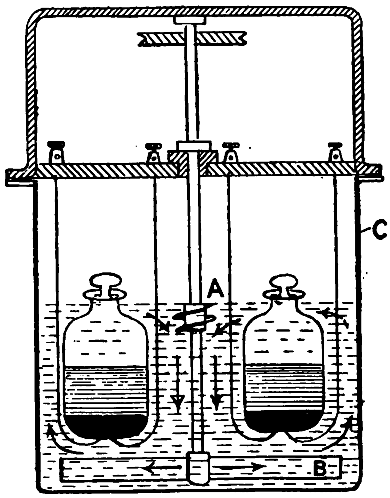
vertical shaft, this being found to be the most satisfactory way of driving them. The ring *h* separates the chamber *g* into two parts: in the upper, the vanes work and the oil is rotating; in the lower, the heating and cooling coils are placed, the oil in this part not rotating. *m* is a distributing plate under the hole in the centre of the ring *h*, which makes the circulating oil pass over the heating and cooling coils.

D is a stationary guide vane which helps to deflect the rotating oil into the next chamber X.

The oil passes under the division *b*, over *c*, under *d*, over *e*, and finally under *f* into the heating-chamber *g*, where it passes over the heating-coil, and is sucked up through the hole in the plate *h*, and, being thrown out by the vanes V, is again circulated.

The circulation obtained by this means was excellent. When the bath was tested at a rate of change of temperature of 1° in eight minutes, there was found to be a maximum difference of temperature between any two points of the bath of 0°·2 ; while at a rate of change of 1° in fifteen minutes, no difference in temperature was detected.

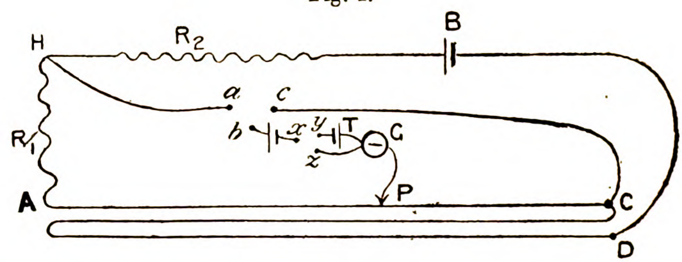
Fig. 3.



In the inner vessel containing the cells (fig. 3) the oil was circulated by means of two sets of vanes ; the upper set, A, in the form of a screw-thread, was arranged so as to send the oil downwards on rotation ; the lower set, B, were radial near the centre, and at the outer ends curved so as to force the oil upwards. Thus, as the oil was forced downwards from the first set A, it was caught, and first sent outwards and then upwards by the second set B, thus producing a constant circulation. During our experiments the temperature in this bath was always uniform to at least 0°·02 C.

The stretched wire (fig. 4), three metres in length, was employed as the potentiometer-wire. The resistance of this being known, the strength of current necessary to give a drop of potential of one ten-thousandth of a volt per centimetre length was calculated, and also the resistance required to be traversed by this current to produce a potential-difference between its extremities equal to the E.M.F. of our standard. The resistance R_1 was adjusted so as to be as nearly as possible equal to that value, and the deficit was found in terms of a length AP of the stretched wire. For example, suppose the resistance necessary to produce a potential-difference equal to the E.M.F. of our standard to be found by calculation to be 214·30 ohms. Plugs would be

Fig. 4.



taken out of R_1 to the value of 214 ohms, and the point P on the wire would be selected such that the resistance of AP equalled 0·30 ohm. Then the mercury cups *a*, *b* and *x*, *z* being joined, the current sent through the wire by the storage-cell B was adjusted by means of the auxiliary variable resistance R_2 until the E.M.F. of the standard equalled the potential-difference between the points H and P. In this way the wire was rendered direct reading for any subsequent comparisons, one centimetre on the scale corresponding with one ten-thousandth of a volt. By joining the mercury cups *a*, *b* and *x*, *z*, the value of the current could be readily checked at any time during the experiment without interrupting the rest of the circuit. As a rule, very little variation was found to occur. In this method we measure the difference of the E.M.Fs. of the cells on the stretched wire, and not the actual E.M.F. of the cell under test; consequently considerable variations of the current have very little effect on the result.

The cell under test was placed at T, and by joining the mercury cups b, c and x, y, this could be joined up with its E.M.F. in opposition to that of the standard ; and then, by moving the contact P, we could find a point such that the potential-difference between P and C equalled the difference in E.M.F. between the cell to be tested and the standard. If the difference between the E.M.Fs. was positive, then the point P lay on one side of the point C, while if the difference was negative, it lay on the other side.

To find how the changes in E.M.F. lagged behind those of the temperature, it was found best to subject the cells to a cycle of temperature, the temperature being first raised at a uniform rate through a certain number of degrees, and then cooled again at the same uniform rate to the original temperature. Supposing no lag to exist, the curve connecting temperature and E.M.F. would be the same for both rising and falling temperatures. But if there is a lag of short duration, *i. e.* not cumulative, then, provided that the temperature varies at the same rate throughout, two distinct curves, the second parallel to the first for the greater part of its length, will result, and will enclose an area between them. If there is any lag of a semi-permanent character, it will be shown by the curve not returning to its original starting-point till after a considerable time.

When we proceeded to take the cells through a cycle of temperature, we first of all determined absolutely the E.M.F. of the standard, which had been kept at a constant temperature for some days in the thermostat. The current for heating the cell-bath was then turned on, the required value of the current to give a certain rate of rise of temperature having been previously determined by experiment. The current in the stretched wire was then adjusted to its proper value, and the readings of the E.M.Fs. of the cells were taken every few minutes. The temperature of the cells was observed every five minutes and a curve plotted connecting temperature and time. The heating curve between time and temperature always gave practically a straight line, without readjustment of the current. When we started to cool our bath, we turned on the iced-water circulation full for the first few minutes, to get a sharp bend in the temperature and time curve, after-

wards it was made to go much slower, and the flow required readjusting from time to time to keep the curve straight. Considering that this adjustment was done purely by the method of trial and error, the curves obtained may be regarded as highly satisfactory. They approximate so closely to straight lines that it was thought unnecessary to reproduce them. In some of the cycles we cooled first and and heated afterwards, thus reversing the cycle. At the conclusion of each experiment the E.M.F. of the standard was again tested absolutely.

The accuracy of the method was all that was desired. The thermostat would keep the temperature of the standard constant to 0°·01 C., and the temperature of the cell-bath was uniform to 0°·02 at least. The difference in the E.M.Fs. could be read on the stretched wire to one hundred-thousandth of a volt. The E.M.F. of the standard never varied more than one ten-thousandth of a volt during an experiment, and in general it probably varied much less.

Three different rates of change of temperature were tried: 1° C. in seven minutes; 1° in fifteen minutes; and 1° in thirty minutes. A cycle generally lasted about five or six hours. The curves are plotted for E.M.F. and temperature, and, to avoid confusion, the curves for the different cells have been separated by displacing the origin of E.M.F. differences.

It will be seen that the curve connecting E.M.F. and temperature is not the same for both the rising and the falling temperatures in a cycle. As explained above, this indicates the existence of lag. The lag of E.M.F. at a rate of change of temperature of 1° in seven minutes is about four ten-thousandths of a volt; at a rate of change of 1° in fifteen minutes, it is about two-and-a-half ten-thousandths; at a rate of 1° in thirty minutes, it is about one twenty-thousandth of a volt. By this lag is meant half the difference between the E.M.Fs. given by the rising and falling curves for any particular temperature. If these lags are compared with those obtained for the B.T.-type of cell, in the paper above referred to, we see that the lag in the case of the H-form of cell is less than one quarter of that in the ordinary form. (See fig. 7. N.B.—The zero-lines on this diagram are purely arbitrary; the curves are merely for the purpose of comparison.)

The temperature at the close of a cycle was sometimes not exactly the same as at the commencement, hence the temperature-time curves are not closed. It is highly probable that if the temperature had been the same, the E.M.F. would have returned to its original value. On keeping the temperature constant at the end of the cycle, the lag passed off rapidly; at a rate of variation of temperature of 1° in fifteen minutes it disappeared in about ten minutes.

The probable reason for the superiority of the H-form over the ordinary form is that when the changes of temperature produce variations in the density of the zinc sulphate solution, the zinc amalgam, being at the bottom, is always in contact with a saturated solution, while in the B.T.-form we have the solution in contact with the mercury saturated, but that in contact with zinc will not be saturated till the whole of the paste has attained its equilibrium condition and again become saturated with zinc sulphate at the new temperature.

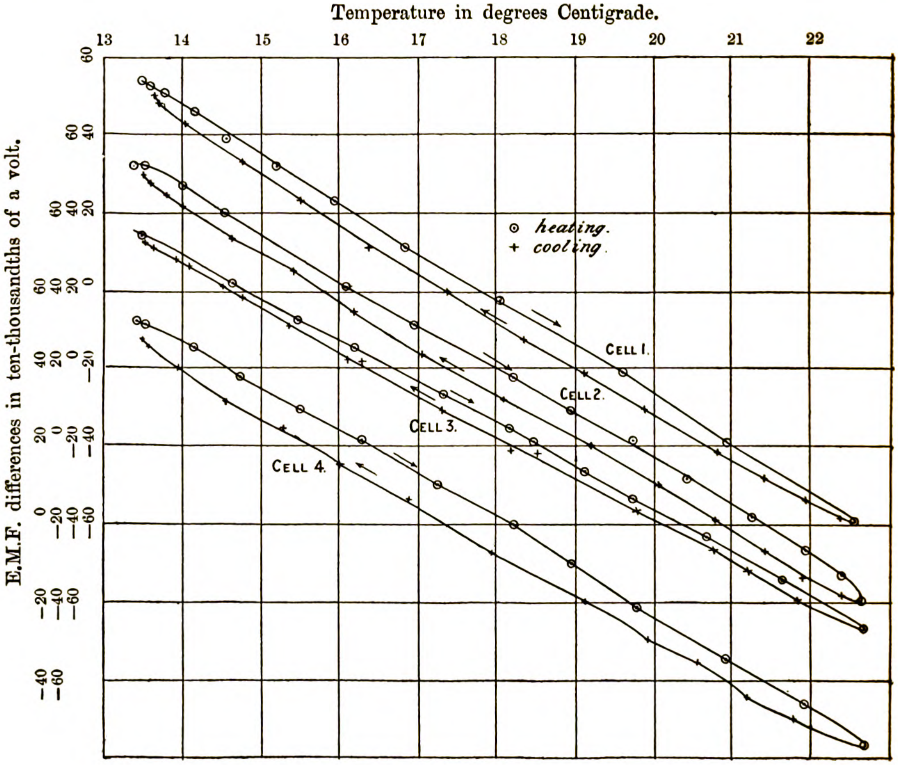
Thus in general it appears that in the H-form of Clark cell the lag is much less than in the B.T.-form, and that under ordinary conditions, when the rate of variation of temperature is not greater than 1° in half an hour, by applying the ordinary temperature-corrections we can obtain a result accurate to less than a ten-thousandth of a volt. If we wish to apply a temperature-correction for a greater rate of variation than this, we must either find the temperature-coefficient for variations at that rate from curves such as those given here, or else slightly diminish the ordinary coefficient. Subsequent experiments showed that if an H-cell were placed in a water-bath, which was heated through 3° or 4° as quickly as possible, then the E.M.F. of the cell would reach an approximately steady value as soon as the temperature of the bath, and by taking the temperature of the cell as being that of the bath, we could apply the temperature correction accurate to a ten-thousandth of a volt.

The two Muirhead cells used as standards in the above experiments were also taken through a single cycle at a rate of variation of temperature of 1° in fifteen minutes. From the curves (fig. 8) it will be seen that one of the cells is but slightly inferior as regards lag to the H-form, while the other comes about midway between the H and the B.T. forms.

These two cells have been recently dismounted, and it was found that they were almost dried up and that the zinc sulphate had crystallized out over the top of the cells. They had thus practically become examples of Prof. Callendar's crystal cell*, and this probably accounts for their small lag. The cell which appeared the drier of the two had the larger lag.

Our best thanks are due to Prof. Ayrton and Mr. T. Mather for their many valuable suggestions and kind help, and to Mr. Cooper for the loan of the cells tested.

Fig. 5.—Curves connecting E.M.F. and Temperature. Rate of Variation of Temperature, 1° in 15 minutes.



* Proc. Roy. Soc. October 1897.

Fig. 6.—Curves connecting E.M.F. a d Temperature; Rate of Change of Temperature 1° in 30 minutes.

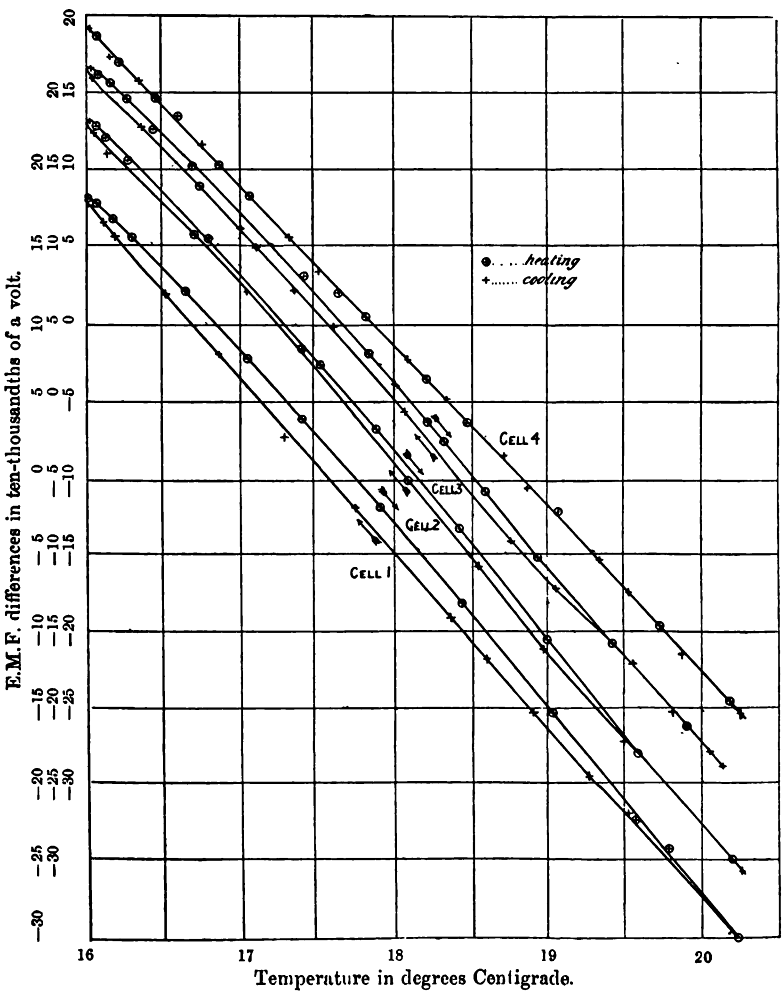
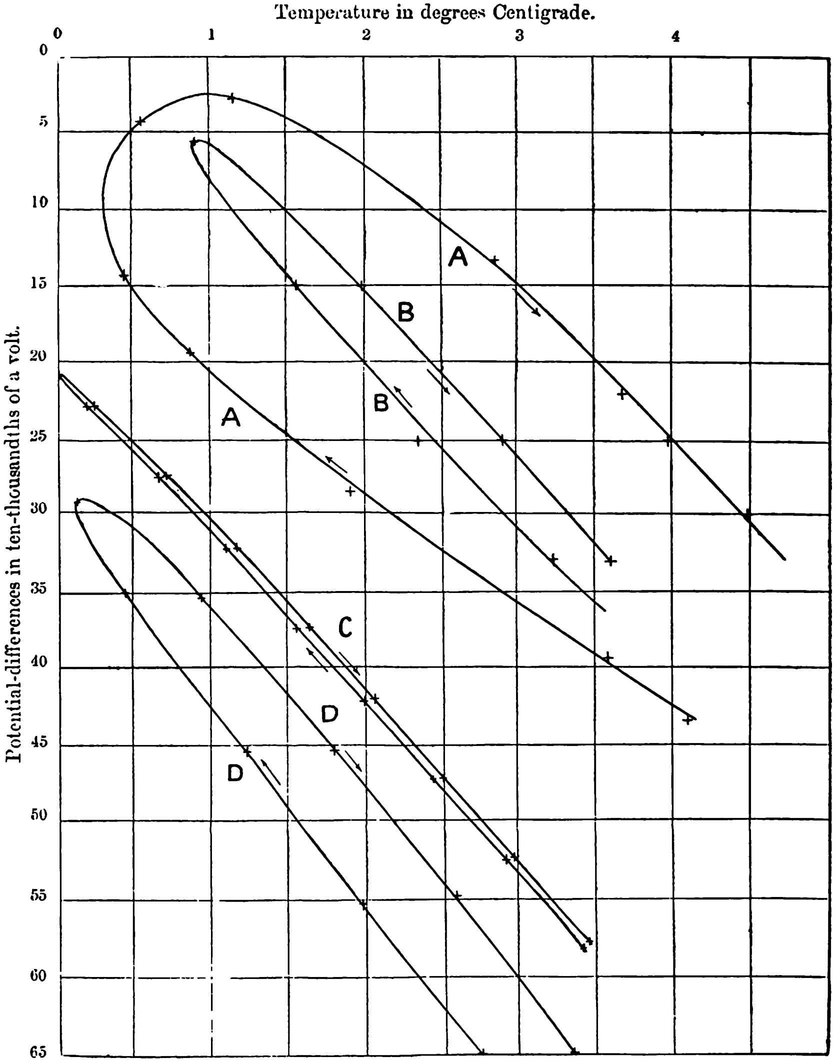


Fig. 7.—Curves comparing the lag in the H-form of cell with that in the B.T.-form.

A. An average curve for Board of Trade cell. Rate of change of temperature, 1° in 15 min. (From paper of Prof. Ayrton and Mr. Cooper.)
B. H-form. 1° in 15 min.
C. H-form. 1° in 30 min.
D. H-form. 1° in 7 min.



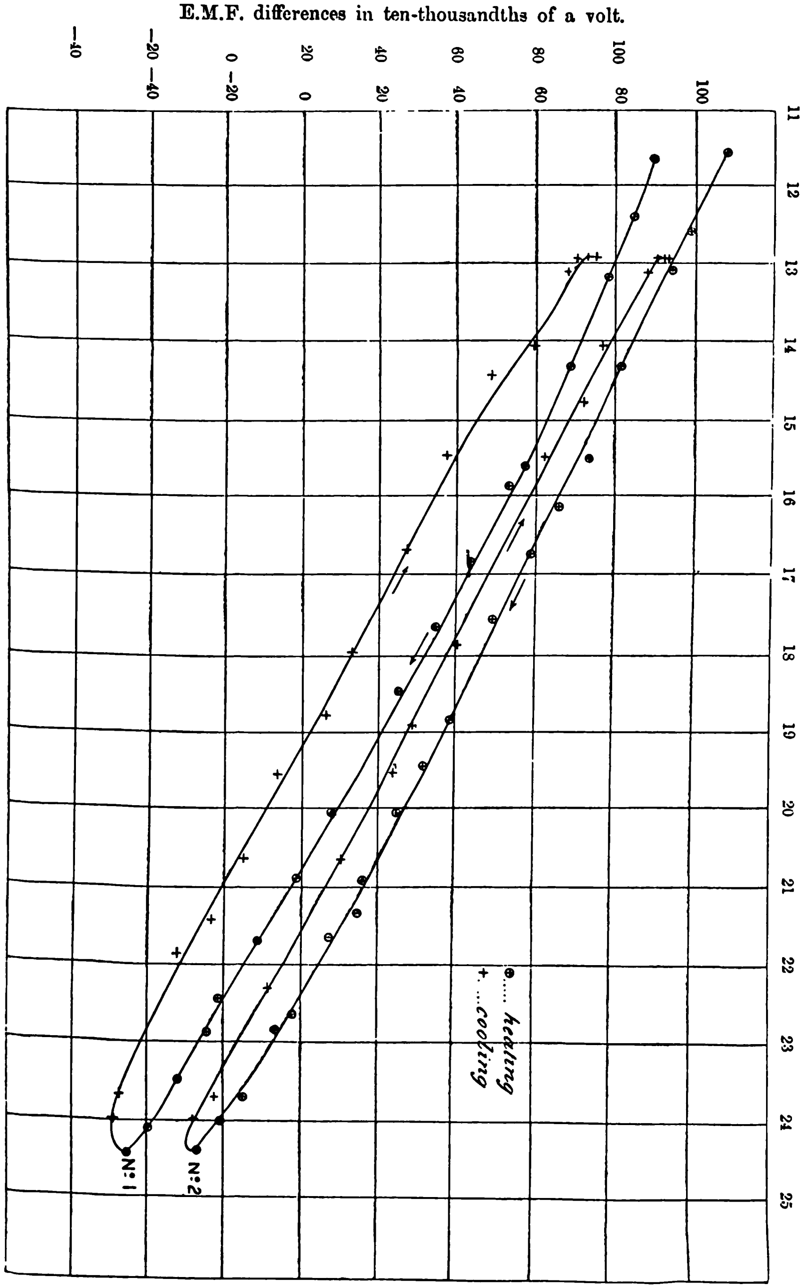


Fig. 8.—Curves of E.M.F. and Temperature. Muirhead Cells. Rate of Variation of Temperature, 1° in 15 minutes.

Appendix.

Since the above investigation was completed, some experiments have been made to test whether the lag of E.M.F. observed was due to the lag of the temperature of the cell behind that of the bath or to diffusion lag. A B.T.-form of cell was made up in a tube 2 cm. diameter, and a thermometer inserted in a position similar to that occupied by the zinc. The cell was immersed in a water-bath so that the upper surface of the paste was about 2 cm. below the surface of the water. The temperature of the bath was then raised uniformly, the bath being kept stirred, and the temperature of the cell observed from time to time and compared with that of the water-bath. The lag of the temperature of the cell behind that of the bath was, at a rate of variation of temperature of 1° in six minutes, 0°·15 ; at a rate of 1° in fifteen minutes 0°·10 ; at a rate of 1° in thirty minutes less than 0°·05. The same experiment was tried with the thermometer half in the mercury and half in the paste so as to represent the condition of things in the H-form. The temperature lags in this position were found to be very slightly less than in the other.

Experiments were also made in which the cell at a temperature of 12° C. was placed in the bath at 30° C. and the bath stirred; the temperature of the cell was observed at different times. The observations showed that at the end of five minutes the lag behind the temperature of the bath was 0°·10, at the end of eight minutes practically zero. The temperatures in these experiments were observed accurate to 0°·05 C.

A Callendar crystal cell was tested in the same manner as the above and gave almost identical results.

It appears from these experiments that about half the lag of E.M.F. observed in the H-form of cell is due to the lag of the temperature of the cell behind that of the bath, and the rest to diffusion lag.

Hence in all experiments with Clark cells, where the temperature effects are likely to be serious, it is better to discard the B.T.-form of cell and use either Prof. Callendar's crystal cell, Prof. Carhart's modification of the B.T.-form,

or the H-form. The Callendar crystal cell, in which the zinc rests in a mass of moist zinc-sulphate crystals, has a very small diffusion lag, but has the disadvantage that there is a possibility of particles of zinc falling and reaching the mercury. This fault is, however, eliminated in his inverted type. The Carhart cell, in which the solution is of constant strength, also possesses a very small diffusion lag, but it has two serious drawbacks, firstly, an error of a few tenths of a degree in the temperature at which the cell is saturated makes an error of several ten-thousandths of a volt in the E.M.F., and, secondly, every time a current passes through the cell a certain amount of zinc is dissolved, so in time the solution becomes stronger. This point is of particular importance where the cells are used with condensers.

Discussion.

Mr. W. R. Cooper thought the authors did not express the case clearly. The E.M.F. of the "Board of Trade" cell was not known to within ·1 per cent. In some cases when, for instance, cells were used differentially, greater accuracy might be required, as, for example, where a constant source of E.M.F. was being compared with the variations of another source. He would like to know with what degree of accuracy the E.M.F. of the standard cell was determined by the authors. The lag that occurred in the "Board of Trade" cell was probably due to diffusion, crystallization, and solution.

Mr. Waters said that the E.M.F. of the standard was measured by a Kelvin balance to one in ten-thousand.

IX. *A Mechanical Cause of Homogeneity of Structure and Symmetry Geometrically Investigated; with special application to Crystals and to Chemical Combination.* By WILLIAM BARLOW.*

[Abstract only. Published *in extenso* in *Scientific Proceedings of the Royal Dublin Society*, vol. viii. (n. s.) part vi., no. 62; also in *Groth's Zeitschrift für Krystallographie und Mineralogie*, xxix. 433, 1898.]

THIS paper is to be regarded as supplemental to the geometrical work on the nature of homogeneity by the same author, by which he has shown that every homogeneous structure, whatever its nature, displays one or other of the thirty-two kinds of crystal symmetry †.

The author begins by alluding to the various artificial devices for packing together a number of similar bodies which have from time to time been employed to imitate the different kinds of symmetry displayed by crystals; also to the evidence afforded by stereo-chemical investigations that a regular repetition in space which portrays the homogeneity of structure of crystals should be that of groups composed of two or more individuals rather than that of single bodies, as generally hitherto represented. He then remarks that probably the simplest conceivable kind of closest packing which gives diversity in unity of the elements of the structure, is that of a large number of spheres of two, three, or more different sizes, and argues that the condition of closest packing leads to homogeneity in a great number of such cases.

In order to avoid arbitrariness he suggests the use of elastic deformable balls, differing in material as well as in size, instead of that of rigid undeformable ones; and he points out that if an assemblage of this kind is subjected to some uniform compression which flattens the balls at the places of contact, it can be regarded as equivalent to a flock of mutually-repellent particles occupying the places of the ball-centres, and in equilibrium; provided that repulsion subsists only between near particles whose interaction is represented by the effect on one another of balls which touch.

* Read March 12, 1897.
† See Mineralogical Magazine, vol. xi. p. 119.

For the purpose of imitating two of the principal universal properties of molecular matter, the balls employed to form the assemblages referred to are (1) regarded as suffering contraction or expansion under change of conditions, those of one size, composed of one material, changing at a different rate from those of another material; and (2) it is postulated that ball can be attached to ball by an almost inextensible tie reaching from centre to centre through the place of contact, so that when tied in this way, the centres of two balls cannot get farther and farther apart under change of conditions, and thus the balls have, so to speak, to interpenetrate between tied centres if they expand.

A considerable number of closest-packed arrangements of balls of one, two, or more different sizes, some of which consist of groups of linked balls, and some of undetached ones, are described, and in each case the type of homogeneous structure which an assemblage presents is indicated, and it is further stated to which of the thirty-two systems of crystal symmetry it belongs.

From these examples it is contended that the following proposition is established :—

That assemblages belonging to all of the thirty-two classes of crystal symmetry result from closest-packing of balls of different sizes, when the relations between the different radii take the widest possible range of variety, and cases of packing together of spheres formed into groups in the way premised are included, as well as the cases in which the spheres are unlinked.

Facts with regard to crystal growth are then cited to show that symmetrical arrangement precedes solidification and that the accretion which takes place in cases of growing crystals is consistent with a conception of closest-packing brought about in the way described.

It is next pointed out that under some conditions the necessity for closest-packing will involve the production of *thin* curved assemblages, and the facts concerning thin curved and branched microscopic and other crystals are compared with cases of curved assemblages originated in this way.

The author then proceeds to describe the effect, in various different assemblages, of changing the conditions to which the balls are subjected past a critical point, and the con-

sequent production of dimorphous assemblages, and compares these results with a number of cases of dimorphism and trimorphism occurring in crystals.

In closing the first section of the paper, it is suggested that a *plane of readiest cleavage* is not necessarily a plane the ties crossing which are ties of minimum strength under any given constant conditions, but a plane on the opposite sides of which contiguous parts which face one another are most diverse.

The second section is devoted to the symmetrical partitioning of the artificial assemblages of balls described in the first section, and to the comparison of the properties of the similar groups or units thus obtained with the experimentally ascertained properties of the molecules of the stereo-chemist, considerable space being given to the discussion of enantiomorphously-similar groupings.

Tables of all the types of grouping possible are added, also of the number of types obtainable by twofold substitution in these typical groupings.

In the third section the author explains the formation of twinned assemblages of balls consistently with closest-packing, and compares the results with the actual crystal twins; the properties of many of the modified artificial assemblages described closely resemble many optical and other anomalies of crystals.

Isomorphism, crystalloid structure, and some kinds of diffusion are also dealt with in this section.

In the sections 5, 6, and 7 certain results of closest-packing which resemble some of the incidents of *chemical change* are referred to.

The paper concludes with an Appendix in which the ideas that form the basis of the investigation are stated as definite concepts, and a fundamental *law of closest-packing* is enunciated.

An index is added.

Discussion.

Prof. Herschel said he was particularly pleased with the models. He thought it probable that a wide application would be found for the author's results. There was no doubt much to be learnt from models built up of spheres of two

or more sizes, but it would be necessary to learn a great deal more about these symmetrical arrangements before they could be applied with any degree of certainty.

Mr. FLETCHER said it was impossible to criticise the paper without long and careful study. From certain hypotheses the author had deduced a law of "closest packing" that seemed adequate to explain many results observed by chemists and crystallographers; at the same time admitting that the law might be presumed from other reasoning. By his models he had tried to present a picture not of the forms of atoms or molecules, but merely analogical representations of the probable structure of particles. Hitherto the research had been confined to determining the possible arrangements of particles all of one kind, but here were examples of packed spheres of various sizes. It was not quite clear how, in an elementary substance, there could be such a structure, although there certainly are cases of polymorphism awaiting explanation, as, for instance, with sulphur. The paper, with its 188 pages of MS., represented a vast amount of clear thinking and many years of admirable work.

Prof. ADAMS called the attention of Fellows of the Physical Society to the Museum at King's College, where were the original models as made and used by the early investigators of this branch of Physics.

Prof. MIERS (communicated too late for reading). The principle of "close packing" was not new, but Mr. Barlow was the first to extend it to explain solution, diffusion, and stereo-chemical problems. His remarks on the growth of curved crystals, vicinal faces, and pseudosymmetrical crystals were open to criticism. With regard to vicinal faces, however, leucite seemed to be a mineral in accord with his hypothesis. The author regarded a crystal as consisting of mutually repellent particles of different sorts; this seemed a very right way of attacking the problem of crystal structure, and would explain some recent observations of Rinne on crystals consisting of water particles and silicate particles. Further, Mr. Barlow had considered the way in which an assemblage might be broken up by the loosening of the ties, and the change of partners, among individual members. That is to say, he had considered crystallization and solution,

features quite ignored by ordinary theories. His view of crystal structure failed to explain why crystals should have faces, and gave no hint as to the controlling forces which keep mutually repellent particles together. Nevertheless it suggested, among other striking analogies, those bearing on the relationship between crystal structure and chemical constitution, and the irregularities of crystals. Mr. Barlow had opened up a very promising line of inquiry.

Mr. BARLOW, in replying, said he greatly appreciated the interest shown in his work.

X. *Electrical Signalling without Connecting-Wires.* By Prof. OLIVER LODGE, *D.Sc., F.R.S., Professor of Physics at University College, Liverpool.**

[Abstract.]

FROM the nature of the oscillatory disturbances emanating from any of the customary forms of Hertz vibrator, syntony has hitherto been only very partially available as a means for discriminating between receivers. There is, in fact, so rapid a decrease in the amplitude of the vibrations that almost any receiver can respond to some extent. Discrimination by syntony is possible with magnetic systems of space telegraphy, where the magnetic energy much exceeds the electric, *i. e.*, as between two separated inductive coils; and by the use of such coils, appropriately applied, the author has been able to attain fair syntony even with true Hertz waves—*i. e.*, he has constructed spark-gap oscillators, with sufficient persistence of vibration, and syntonised resonators. The "coherer" principle can be applied to either a purely magnetic or to the Hertzian system. It was first used by the author in devising lightning-guards, and afterwards in his magnetic system of telegraphy by inductive circuits, each in series with a Leyden jar; a pair of knobs in near contact, or other over-flow gap, being provided in the receiving apparatus. This was the first meaning of a "coherer" in the electrical sense as used by the author: it referred to a

* Read January 21, 1898.

single contact between two metal knobs. The term has since been extended by others to the filings-tube of M. Branly, and some confusion has arisen, for M. Branly does not consider that simple coherence and break explains fully the behaviour of his instrument. The author is disposed to agree, for he finds that the resistance of almost any form of coherer varies in rough proportion to the received impulses, and that there are other peculiarities (to be mentioned later); he is therefore inclined to think that the action cannot after all be entirely explained as due to mere "welding," but that there is something more to be learnt about it. The sensitiveness of a coherer depends upon the number of loose contacts: it is a maximum for a single contact, *i. e.*, for a needle-point lightly touching a steel spring. With this sensitive coherer, hardly any "tapping-back" is required for decoherence, but it wants delicate treatment when properly adjusted, and the greatest current through it should not approach a milliampere. On the other hand, a Branly tube rather improves under rough treatment; in such a tube the author prefers to use iron filings in the best possible vacuum: brass, too, is very good, but rather less easy to manage. Aluminium is thoroughly bad, and gold, for an opposite reason, will not work—its surface is too clean. Points, or small surfaces for making contact with the filings, are better than large surfaces. The usual method of connecting the coherer across the gap of an ordinary Hertz receiver, in parallel with the telegraph instrument and battery, has the unavoidable objection that they shunt away part of the received oscillations. With the syntonic receiver which contains no gap but a closed wire coil instead, the difficulty no longer exists, for the coherer can now be in series with the detecting instrument; and, in so far as these obstruct the oscillations, they may be shunted out in various ways, as the author describes. The main feature of his new syntonised vibrators is this self-inductance coil, whose function it is to prolong the duration of the oscillations, and thereby to render syntony possible. Although such a coil acts disadvantageously in so far as it possesses resistance, the resistance does not increase so fast as the self-induction. The coil should consist of stranded copper of highest conductivity, and it should have maximum inductance for given resistance.

For similar reasons, the capacity areas should also be of highest conductivity, their dimensions should increase outwards from the spark gap, as triangles. The receiver must have no gap, it should be accurately bridged over when a transmitter is used as receiver. The limit of speed of response depends upon the telegraphic instrument. Dr. Muirhead adapted a syphon-recorder to the purpose, because it is one of the quickest responders; he arranged it so that it could be used with intermittent currents, direct. Under these intermittent impulses the syphon moves and records the fluctuations of current on the paper; or it can be arranged to tremble so that the slip is marked with dots and dashes. Constant mechanical tremor is usually employed for decoherence, but the author finds that decoherence can be brought about by electrical means, without any mechanical tremor, by connecting the coherer momentarily to a circuit less effective as a collector than that of the proper capacity areas of the syntonised receiver. The battery and galvanometer detector-circuit may be used for this purpose, the coherer being momentarily connected to it, and while so connected letting it experience an impulse from a distance. The author has designed a revolving commutator, by means of which the coherer can be rapidly changed over from the resonating circuit to the instrument circuit, and finally to the "tapping-back" apparatus. A coherer is more sensitive when thus isolated and exposed to the full influence of the received oscillations; the subsequent detection of the effect by altered connections is very convenient for laboratory measurements. A diagram of a series of plotted measurements showed that the resistance of an undisturbed filings-tube is approximately a direct function of the intensity of the received stimulus, whether successive stimuli increased or decreased in strength. This electrical process of "tapping back" is dependable for a time, but the process if long continued fatigues the tube until a mechanical shake is employed to restore it. A large apparatus, made by Dr. Muirhead for actual distant syntonic work, was exhibited, and means were shown for protecting it and isolating the coherer when its own receiving areas were being used as emitters; also a special switch which is used for changing at one movement all the connections from "sending" to "receiving."

Discussion.

Prof. Threlfall said he had come to the same conclusion as Prof. Lodge as to the advisability of diminishing the number of contact points in the coherer. He had endeavoured to produce longer and more persistent waves, and thus to set afield greater effective energy. It was desirable to keep the waves as parallel as possible. He thought there was some probability that the wave-fronts could be altered and rendered more conformable by a process of diffraction.

Mr. Rutherford also had found it best to work with long waves. He fully appreciated the advantage of increasing the capacity of the oscillator by extending the surface of the metallic plates.

Mr. Campbell-Swinton asked whether experiments had been made to verify Hertz's results as to the influence of reflectors behind oscillators and receivers. He had found them disadvantageous. A single wire behind either apparatus seemed partially to annul the effect. He also asked whether Prof. Lodge had observed the extraordinary sensitiveness of coherers to small changes of current in neighbouring circuits.

Prof. Lodge, in reply, said he had observed the sensitiveness to slight sudden variations of current referred to by Mr. Campbell-Swinton ; for instance, when electric lamps were switched on and off. The effect of mirrors had been studied by Prof. Fitzgerald. They required to be of large dimensions as compared to the oscill itor and receiver, otherwise the true reflections were not obtained.

XI. *Exhibition of a Tesla Coil.* *By* Prof. Silvanus Thompson, *D.Sc.*, *F.R.S.**

This apparatus is intended to replace the two induction-coils and spark-gap arrangements used by Mr. Tesla for high-frequency experiments. It consists of an induction-coil, with a separate self-inductance coil in the primary circuit. This self-inductance coil is also used as an electromagnet for the separate interrupter of the primary circuit. A condenser of 2·27 microfarads capacity is connected between one end of

* January 21, 1898.

the primary coil and one terminal of the interrupter, so as to include both of them between its terminals. The primary consists of 5 turns of copper strip. The secondary has 2000 turns of thick wire. The supply current, about half an ampere, may be taken from the electric-light mains at almost any voltage from 50 to 200, direct or alternating.

Discussion.

Prof. Lodge said that he believed Sir W. Crookes had found it work quite well at 10 volts. He pointed out also that if the straight discharge rods at the spark-gap were free to slide, the discharge drove them back into their sockets.

Prof. Fitzgerald said it was stated at Toronto that the spark was broken at the interrupter when the condenser was charged, and that by the time the condenser was ready to discharge, the contact at the interrupter had been made again. It had seemed to him at first that the condenser discharges and surgings must take place at a rate far higher than the period of the mechanical movement of the interrupter; but he perceived afterwards that it was not so.

Prof. Herschel asked if such an apparatus was suitable for work with Röntgen rays.

Dr. Thompson, in reply, congratulated Mr. Tesla upon the perfect working and compactness of his invention. The present form was not suited for Röntgen ray experiments, with ordinary focus-tubes, but Mr. Tesla had designed a special tube that was excellent for that purpose.

XII. *Winter Observations on the Shadow of the Peak of Tenerife, with a new Method for measuring approximately the Diameter of the Earth. By* Rev. T. C. Porter, *M.A.**

[Abstract.]

The method consists in observing the shadow cast by the Peak upon the sea, and measuring the time that elapses between the moment when the apex of the shadow touches

* Read February 26, 1898.

the sea-horizon, and the instant when it is eclipsed by the shadow of night.

The author calls attention to a phenomenon hitherto unnoticed, *i. e.*, that the heated air ascending from the Peak casts a shadow, seen as a faint prolongation of that of the Peak, rising obliquely from its apex. A photograph was exhibited, taken on a quarter-plate, in which is visible the curvature of the horizon as viewed from the altitude of the Peak.

An interesting series of photographs, illustrating the conformation of the Peak and the phenomena of sunrise and twilight in that latitude, was also shown. In regard to twilight, it is noticed that the first approach of night, as observed looking eastward, is marked by a dark border of about five degrees width, followed by a sky somewhat lighter.

XIII. *A new Theory of Geysers.* *By* Rev. T. C. Porter, *M.A.**

[Abstract.]

THE theories of Bunsen and others fail to explain why the geyser throat appears almost completely full at the end of an eruption. This immediate re-filling is the more remarkable when it is remembered that some of the geysers of the Yellowstone region discharge a million and a half gallons at each eruption, and that the eruptions may occur at five-minute intervals. Moreover, the theories generally accepted assume steeper temperature-gradients than those in a region like Yellowstone. The author suggests that the phenomena are better explained on the assumption of an arrangement of strata such as exists in artesian-well districts ; the throat or shaft of the geyser being in the position of a well communicating with a subterranean stream—the "tube" of the geyser. From the disturbed nature of the region, the tube of the geyser probably follows a waved course. The "shaft"

* Read February 26, 1898.

rises from the crest of the terminal wave; the other crests may act as steam-traps. Since a basin-like formation is characteristic of all geyser regions, it is fair to assume that the end of the tube remote from the shaft has an outcrop in the hills that form the sides of the basin. By means of this outcrop, water continually flows into the tube. Where the tube does not sink deeply enough to attain the temperature necessary for the generation of steam, a quietly-flowing hot spring is the result. But if at any point the tube descends to underground temperatures sufficiently great, steam is formed, and is trapped at the highest point of a bend. Ultimately this steam checks the flow of water, until the accumulated head of cool water from the hills overcomes the resistance, condenses the steam, and re-establishes liquid continuity. Urged by the pressure behind it, the steam is impelled towards the geyser throat; it forces the hot water before it until equilibrium is once again restored in the tube.

XIV. *A Method of Viewing Lantern Projections in Stereoscopic Relief.* *By* Rev. T. C. Porter, *M.A.**

[Abstract.]

A slotted disc rotates in front of two lanterns. These project two stereoscopic views in rapid alternation upon a screen in such a way that the two projections are approximately superposed. In the rim of the disc other slots are cut, through which the observer looks. The arrangement of slots is such that the right or left eye is only able to see the screen at the moment when its own picture, *i. e.*, the picture from the right or left lantern, is on the screen. When the rotation is sufficiently rapid, the views appear as one, without "flicker," in stereoscopic relief.

* Read February 26, 1898.

XV. *On Electromagnetic Induction in Plane, Cylindrical, and Spherical Current-Sheets, and its Representation by Moving Trails of Images.*—Parts II. & III. *By* G. H. BRYAN, *Sc.D., F.R.S.**

Part II. CYLINDRICAL SHEETS.

IN the Proceedings of the Physical Society for 1894 (vol. xiii. p. 145), I showed how to obtain in a simple way the surface-conditions satisfied by the magnetic potential at a thin sheet of metal of finite conductivity of any form, in which currents are induced by the motion of magnets on either side of the sheet.

In the case of a *cylindrical* sheet of small thickness c, conductivity C, and radius a, the magnetic potentials Ω_1 and Ω_2 inside and outside the sheet were found to be connected by surface-conditions (at $r=a$) which may be written

$$\frac{d}{dt}\left(r\frac{d\Omega_1}{dr}\right)=\frac{d}{dt}\left(r\frac{d\Omega_2}{dr}\right)=\frac{\mathrm{R}}{2}\frac{d}{dr}r\frac{d}{dr}(\Omega_2-\Omega_1), \quad . \quad . \quad (1)$$

involving the condition $d\Omega_1/dr=d\Omega_2/dr$; where

$$\mathrm{R}=\frac{1}{2\pi \mathrm{C}c},$$

so that R is the velocity with which the trails of images in a plane sheet of the same thickness and conductivity would move away from the sheet.

For a *spherical* sheet we found in like manner

$$\frac{d}{dt}\left(r^2\frac{d\Omega_1}{dr}\right)=\frac{d}{dt}\left(r^2\frac{d\Omega_2}{dr}\right)=\frac{\mathrm{R}}{2}\frac{d}{dr}r^2\frac{d}{dr}(\Omega_2-\Omega_1), \quad . \quad (2)$$

involving, as before, the condition $d\Omega_1/dr=d\Omega_2/dr$.

Before determining the images which represent the effect of the currents induced by the introduction of a given pole, magnet, or current in the presence of such sheets, it will be convenient to think, for a moment, of the images which occur in hydrodynamical problems relating to the cylinder and sphere. In two-dimensional fluid motion, the image of a line-source outside and parallel to the axis of a cylinder is an equal

* Read February 11, 1898.

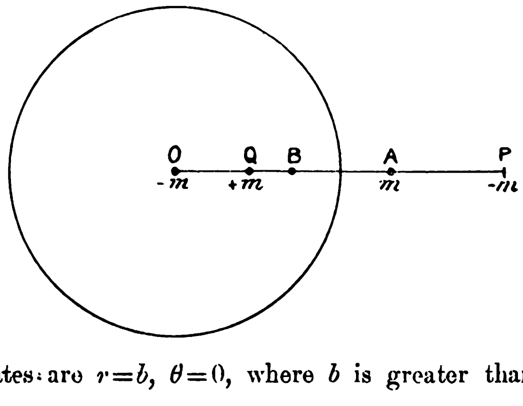
line-source through the inverse point together with an equal line-sink through the centre (*i. e.* along the axis); similarly the image of a line-source inside the sphere together with an equal line-sink along the axis is an equal line-source through the inverse point. The image of a vortex-filament parallel to the axis of a cylinder, whether inside or outside the cylinder, is an equal and opposite vortex through the inverse point. In the case of a sphere, the image of a doublet whose axis passes through the centre is a certain doublet at the inverse point, whether the original doublet be inside or outside the sphere. The image of a *source* in a sphere, on the other hand, is more complicated, consisting not only of a source but also of a sink extending to the centre if the original source was outside the sphere, and to infinity if inside.

We may therefore expect by analogy that the images in a spherical sheet due to the motion or generation of magnetic *poles* in its neighbourhood will be less simple than those produced by *magnets* whose axes pass through the centre of the sphere; and such we shall find to be actually the case.

Image due to Generation of a Line of Poles parallel to the Axis outside a Cylindrical Sheet.

2. We first consider the two-dimensional problem in which the currents are induced by the generation or destruction of a linear distribution of magnetism parallel to the axis of the cylinder at A (fig. 1) in the line whose polar (cylindrical)

Fig. 1.



coordinates are $r=b$, $\theta=0$, where b is greater than a the

radius of the cylinder. To cover all cases we commence by assuming at the outset that the line-density of this line of poles, as it may be called, is $f(t)$ a function of the time.

The magnetic potential of this distribution at the point (r, θ) is therefore

$$\Omega_0 \text{ (say) } = \mu f(t)\{\text{const.} - \log \sqrt{(r^2+b^2-2rb\cos\theta)}\},$$

where μ, the magnetic permeability of the dielectric, will be taken to be unity, or included under $f(t)$.

The above potential may, for values of r less than b, be expanded in the form

$$\Omega_0 = f(t)\left\{\text{const.} - \log b + \Sigma \frac{1}{n}\frac{r^n}{b^n}\cos n\theta\right\}. \quad . \quad (3)$$

Consistently with the surface condition of continuity of magnetic induction, we can assume for the potentials of the induced currents inside and outside the cylinder expressions of the form

$$\Omega' = -\Sigma \frac{A_n}{n}\frac{r^n}{a^n}\cos n\theta \text{ (inside)}, \quad . \quad . \quad . \quad (4)$$

$$\Omega'' = \Sigma \frac{A_n}{n}\frac{a^n}{r^n}\cos n\theta \text{ (outside)}, \quad . \quad . \quad . \quad (5)$$

since these make $d\Omega'/dr = d\Omega''/dr$ when $r=a$; here the coefficients A_n are functions of the time t.

Substituting the expressions thus obtained, $\Omega_0+\Omega'$ for Ω_1, and $\Omega_0+\Omega''$ for Ω_2 in the surface-conditions (1), we have, on equating coefficients of $\cos n\theta$,

$$\frac{dA_n}{dt} + n\frac{R}{a}A_n = f'(t)\frac{a^n}{b^n};$$

$$\therefore \quad \frac{d}{dt}\left\{A_n e^{ntR/a}\right\} = f'(t)e^{ntR/a}\frac{a^n}{b^n};$$

$$\therefore \quad A_n = \int_{-\infty}^{t} f'(\tau)e^{-n(t-\tau)R/a}d\tau \, . \, \frac{a^n}{b^n},$$

the limits of integration being t and $-\infty$ if we have to take account of changes that have been taking place from an infinite time back.

To analyse the images, consider now the case in which a

line of poles of strength m is suddenly generated at A at the time 0, so that

$$f(t) = 0 \text{ when } t < 0,$$
$$f(t) = m \text{ when } t > 0,$$

$f'(t)$ vanishes except when $t=0$, and the time-integral of $f'(t)$ over a very small interval containing the time 0 is equal to m. We thus obtain

$$\mathrm{A}_n = me^{-nt\mathrm{R}/a}\frac{a^n}{b^n}.$$

Substituting in the expressions for Ω' and Ω'' we find

$$\Omega' = -m\Sigma\frac{1}{n}\frac{r^n}{b^n e^{nt\mathrm{R}/a}}\cos n\theta$$
$$= -m[\log be^{t\mathrm{R}/a} - \log\{r^2 + (be^{t\mathrm{R}/a})^2 - 2rbe^{t\mathrm{R}/a}\cos\theta\}^{\frac{1}{2}}] \quad . \quad (6)$$

$$\Omega'' = m\Sigma\frac{1}{n}\frac{a^{2n}}{b^n e^{nt\mathrm{R}/a}}\cos n\theta$$
$$= m\left[\log r - \log\left\{r^2 + \left(\frac{a^2}{be^{t\mathrm{R}/a}}\right)^2 - 2r\left(\frac{a^2}{be^{t\mathrm{R}/a}}\right)\cos\theta\right\}^{\frac{1}{2}}\right]. \quad (7)$$

Hence we have the following results :—

(i.) At points inside the conductor, the magnetic potential due to the induced currents initially neutralizes that produced by the sudden generation of the line of poles at A, and is equivalent to that of a line of poles (P, fig. 1) of constant strength $-m$, starting at the point A and moving away from the centre so that its distance from the centre at time t is $e^{t\mathrm{R}/a}b$.

(ii.) At points outside the conductor the magnetic potential is equivalent to that of two lines of poles, one of constant strength $-m$ at O the centre, and the other (Q) of strength $+m$, starting at B the inverse point of A $(a^2/b, 0)$, and moving towards the centre so that at time t its distance from the centre is $e^{-t\mathrm{R}/a}a^2/b$. We observe that P and Q, the positions of the inside and outside images, move so as always to remain inverse points.

The velocities with which the images P, Q move away from and towards O respectively are R/a times their distances from O, and their accelerations are R^2/a^2 times these distances respectively.

3. The current function at any point M of the cylinder is

given by

$$\phi = \frac{1}{4\pi}(\Omega_2 - \Omega_1) + \text{a constant};$$

therefore ϕ differs by a constant from

$$-\frac{m}{4\pi}(\log \text{PM} + \log \text{QM}).$$

The arbitrary constants in ϕ may be functions of the time, but they do not affect the distribution of currents in the cylinder, and therefore need not be considered. Since P and Q are inverse points

$$\text{PM} : \text{QM} = \text{OP} : a = a : \text{OQ} = be^{Rt/a} : a;$$

log PM and log QM therefore differ by a term which may be included in the arbitrary constant of ϕ, so that for determining the currents we may take

$$\phi = -\frac{m}{2\pi}\log \text{PM}, \quad \text{or } \phi = -\frac{m}{2\pi}\log \text{QM},$$

whichever is most convenient.

The current at M is evidently parallel to the axis (since ϕ is independent of the third cylindrical coordinate z), and the intensity of the current is equal to

$$\frac{d\phi}{ad\theta} = -\frac{m}{2\pi}\frac{\text{OP}\sin\theta}{\text{PM}^2} = -\frac{m}{2\pi}\frac{\text{OQ}\sin\theta}{\text{QM}^2}.$$

Images due to Generation of Lines of Poles inside the Cylindrical Sheet.

4. If we were to suppose the currents to be induced by a line of poles of strength $f(t)$ at the point $(b, 0)$ where $b < a$, we should have to expand the potential in descending powers of r in order to obtain a convergent series at the surface of the sheet. We should thus obtain for the inducing potential

$$\Omega_0 = f(t)\left\{\text{const.} - \log r + \Sigma\frac{1}{n}\frac{b^n}{r^n}\cos n\theta\right\}.$$

Now the portion

$$f(t)\{\text{const.} - \log r\}$$

represents the potential of a line of poles of strength $f(t)$ along the axis, for investigating the effects of which the

image method is inconvenient. To remove these terms we shall suppose the inducing system to consist of a line of poles through the point $(b, 0)$ together with an equal and opposite line of poles along the axis: in other words of a line of magnets having their positive poles $f(t)$ at $(b, 0)$ and their negative poles $-f(t)$ at the axis. The potential of such a distribution expands where $r > b$ in the form

$$\Omega_0 = f(t) \Sigma \frac{1}{n} \frac{b^n}{r^n} \cos n\theta. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The analysis is almost identical with that of the preceding case, and passing to the case when

$$f(t) = 0 \text{ when } t < 0,$$
$$f(t) = m \text{ when } t > 0,$$

the following results may be obtained, taking B (fig. 1) to now represent the point $(b, 0)$.

(i.) If lines of poles of strengths m and $-m$ be instantaneously generated at B and the axis respectively, the images representing the magnetic potential of the induced currents outside the cylinder after a time t are lines of poles of strengths $-m$ and $+m$ at Q and the axis, where $OQ = e^{-tR/a}b$. Hence the *total* magnetic potential at outside points due to the poles and the induced currents at time t is that of lines of poles of strengths $+m$ and $-m$ at B and Q respectively.

(ii.) At points inside the cylinder, the magnetic potential due to the induced currents alone is that which would be produced by a line of poles of strength $+m$ at P, where P is the inverse point of Q, so that $OP = e^{tR/a}a^2/b$.

Images due to Generation of Straight Currents Parallel to the Axis of the Cylindrical Sheet.

5. The cases where the currents are induced by the sudden generation of a straight current parallel to the axis of the cylinder probably have a more practical physical interest than those considered in the previous articles. If we suppose all the lines of poles of strengths $\pm m$ of the preceding sections to be replaced by currents of intensities $\pm I$ the results regarding the images will be identically the same as before. To satisfy ourselves that such is the case we have only to notice that while the magnetic potential of a line

of poles of line-density m at $(b, 0)$ is the real part of the expression

$$m \log (re^{i\theta} - b),$$

the magnetic potential of a current perpendicular to the plane of reference and of intensity $\frac{1}{2}m$ is the imaginary part of the same expression. Where $r < b$, the potential of the current expands therefore in the series

$$m\Sigma \frac{1}{n} \frac{r^n}{b^n} \sin n\theta,$$

and where $r > b$ the potential of a current $\frac{1}{2}m$ at $(b, 0)$ together with a return current $-\frac{1}{2}m$ down the axis expands in the series

$$m\Sigma \frac{1}{n} \frac{b^n}{r^n} \sin n\theta.$$

These expressions differ from those used in the previous articles by the substitution of sines for cosines only, and it follows that when we make similar changes in the assumed expressions for the potentials of the induced currents, the work of substituting in the boundary conditions will be unaltered. For convenience, however, it will be well to again state the final conclusions, which are as follows :—

(i.) If a current of strength I be suddenly generated parallel to the axis of the cylinder at A *outside* the cylinder, the image representing the effect of the induced currents for inside points at time t will be a current of strength $-$I at P, and the images representing their effect for outside points will be a current $+$I at Q and a return current $-$I in the axis, where $\mathrm{OP} = e^{t\mathrm{R}/a}\mathrm{OA}$, and Q is the inverse point of P, so that $\mathrm{OQ} = e^{-t\mathrm{R}/a}a^2/\mathrm{OA}$.

(ii.) If a current of strength I be suddenly generated at B *inside* the cylinder, accompanied by a return current $-$I in the axis, the image representing the effect of the induced currents for outside points will be a current I at P, and the images representing their effect for inside points will be currents $-$I at Q, and I in the axis, where $\mathrm{OQ} = e^{-t\mathrm{R}/a}\mathrm{OB}$, and P is the inverse point of Q, so that $\mathrm{OP} = e^{t\mathrm{R}/a}a^2/\mathrm{OB}$.

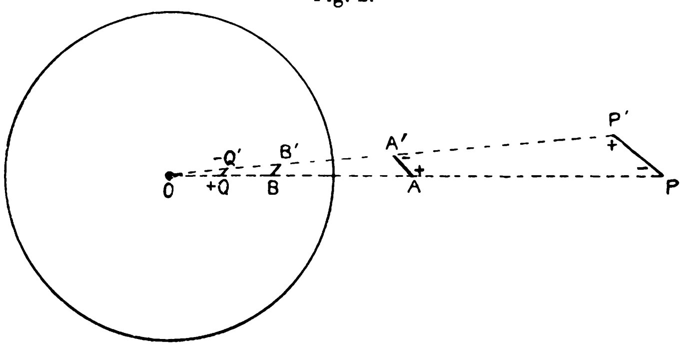
The effect of *breaking* a current of intensity I will of course be the same as that of superposing a current $-$I and the images will be equal and opposite to those above obtained.

Images due to sudden Generation of Lines of Magnets, and Moving Poles.

6. It is now easy to build up the images due to the sudden generation or destruction of lines of magnets parallel to the axis of the cylinder, the motion of lines of poles and so forth. The methods are almost identical with those used for plane sheets, and it will be sufficient here to call attention to one or two salient features of the images in cylindrical sheets. As in the previous articles we shall take A, B to be any two inverse points inside and outside the cylinder, P and Q two other inverse points such that $\mathrm{OP}=e^{t\mathrm{R}.a}\mathrm{OA}$, $\mathrm{OQ}=e^{-t\mathrm{R}.a}\mathrm{OB}$, and similar relations will be supposed to hold when the letters are accented.

(i.) If a line of magnets parallel to the axis at AA′ be generated suddenly, having their positive poles at A and their negative poles at A′ (fig. 2), the image at time t for inside points will be the oppositely magnetized line PP′, and for outside points the line QQ′ having its positive magnetism at Q. From the properties of similar triangles and of inversion, PP′ is

Fig. 2.



parallel and QQ′ is antiparallel to AA′, also, since the strengths of the poles are constant, the magnetic moments of the images are to one another and to that of the inducing system as PP′ : QQ′ : AA′, that is as OP : OQ′ : OA or, in the limiting case (when Q and Q′ are infinitely near together), as the distances OP : OQ : OA.

Hence if M be the magnetic moment generated in AA′,

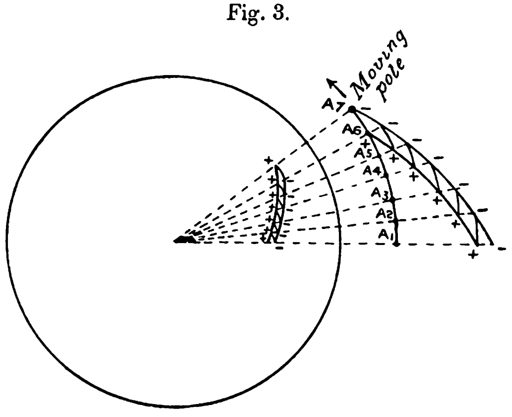
those of the images PP′ and QQ′ at time t are $-\mathrm{M}e^{t\mathrm{R}/a}$ and $\mathrm{M}e^{-t\mathrm{R}/a}a^2/b^2$, where $b=\mathrm{OA}$.

(ii.) Similarly, if a line of magnets of moment M be suddenly generated at BB′ *inside* the cylinder, the images at time t for outside points will be the oppositely magnetized line QQ′ of magnetic moment $-\mathrm{M}e^{-t\mathrm{R}/a}$, and for inside points the magnetized line PP′ having its positive poles at P and of magnetic moment $\mathrm{M}e^{t\mathrm{R}/a}a^2/b^2$, b being now taken $=\mathrm{OB}$.

7. If a line of poles of strength m suddenly changes position from A′ to A the effect is equivalent to the superposition of $-m$ at A′ and $+m$ at A, and is thus equivalent to the generation of a line of magnets in the direction AA′, the images of which have just been found.

Fig. 3 shows the application of this method to construct the trails of images of a line of poles revolving uniformly about the centre and passing in succession through the positions A_1, A_2, A_3, A_4, ..., at equal small intervals of time, the actual trails are of course the limits of the series of moving poles thus plotted out. By supposing the line of poles to

Fig. 3.



remain fixed and the cylinder to rotate, we get a kind of cylindrical two-dimensional Arago's disk, and it is to be noticed that the trails of images lie along equiangular spirals (as may readily be verified), the intensity of magnetization of the trail decreasing toward the centre and increasing outward and being, in fact, proportional to the distance from the centre.

Part III. Spherical Sheets.

Image due to Generation of a Small Magnet whose Axis is Radial outside a Spherical Sheet.

8. We now consider the problem presented by the currents induced in a spherical sheet of radius a when a magnet is generated at a distance b from the centre, the axis of the magnet passing through the centre of the sphere. Taking this axis as the axis of zonal spherical harmonics, and starting in the first instance with a magnet of variable moment $f(t)$, (the positive pole being furthest from the centre), the magnetic potential Ω_0 at a point (r, θ) is given by

$$\Omega_0 = f(t)\frac{d}{db}\frac{1}{\sqrt{(r^2+b^2-2rb\cos\theta)}}.$$

When $r<b$ this becomes

$$\Omega_0 = f(t)\frac{d}{db}\sum_{n=0}^{n=\infty}\frac{r^n}{b^{n+1}}\mathrm{P}_n(\cos\theta)$$

$$= -f(t)\sum_{n=0}^{n=\infty}\frac{(n+1)r^n}{b^{n+2}}\mathrm{P}_n; \quad . \quad . \quad . \quad (9)$$

and when $r>b$ we have in like manner

$$\Omega_0 = f(t)\frac{d}{db}\sum_{n=0}^{n=\infty}\frac{d}{db}\frac{b^n}{r^{n+1}}\mathrm{P}_n(\cos\theta)$$

$$= f(t)\sum_{n=1}^{n=\infty}\frac{nb^{n-1}}{r^{n+1}}\mathrm{P}_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Considering the case where the magnet is outside the sphere, we must take the first expression, since at the surface of the sphere $r<b$.

Let the potentials due to the induced currents at points inside and outside the sphere be assumed of the form

$$\Omega' = \Sigma(n+1)\mathrm{A}\frac{r^n}{a^{n+1}}\mathrm{P}_n \text{ (inside)}, \quad . \quad . \quad (11)$$

$$\Omega'' = -\Sigma n\mathrm{A}_n\frac{a^n}{r^{n+1}}\mathrm{P}_n \text{ (outside)}, \quad . \quad . \quad (12)$$

where the coefficients A_n are functions of the time t; the assumed forms being such as to satisfy the surface-condition $d\Omega'/dr = d\Omega''/dr$.

Substituting the expressions for the potential, $\Omega_0+\Omega'$ and $\Omega_0+\Omega''$ for Ω_1 and Ω_2 in the surface-conditions (2),

$$\frac{d}{dt}\left(r^2\frac{d\Omega_1}{dr}\right)=\frac{d}{dt}\left(r^2\frac{d\Omega_2}{dr}\right)=\frac{R}{2}\frac{d}{dr}r^2\frac{d}{dr}(\Omega_2-\Omega_1),$$

we find, on equating coefficients of P_n,

$$\frac{dA_n}{dt}+\frac{(2n+1)R}{2a}A_n=f'(t)\frac{a^{n+1}}{b^{n+2}},$$

$$\therefore\quad \frac{d}{dt}\{A_n e^{(2n+1)tR/2a}\}=f'(t)e^{(2n+1)tR/2a}\frac{a^{n+1}}{b^{n+2}},$$

$$\therefore\quad A_n=\int_{-\infty}^{t}f'(\tau)e^{-(2n+1)(t-\tau)R/2a}d\tau\cdot\frac{a^{n+1}}{b^{n+2}}.$$

Now let the inducing system be a magnet of moment M suddenly generated at the time $t=0$, so that

$$f(t)=0 \text{ when } t<0,$$
$$f(t)=M \text{ when } t>0.$$

Then by a similar artifice to that used for the cylindrical sheet, we have

$$A_n=Me^{-(2n+1)tR/2a}\frac{a^{n+1}}{b^{n+2}}.$$

Substituting in the potentials of the induced currents, we have

$$\Omega'=M\Sigma(n+1)e^{-(2n+1)tR/2a}\frac{r^n}{b^{n+2}}P_n$$

$$=Me^{3tR/2a}\Sigma\frac{(n+1)r^n}{(e^{tR/a}b)^{n+2}}P_n, \quad .\ .\ .\ .\ (13)$$

$$\Omega''=-M\Sigma ne^{-(2n+1)tR/2a}\frac{a^{2n+1}}{b^{n+2}r^{n+1}}P_n.$$

$$=-M\frac{a^3}{b^3}e^{3tR/2a}\Sigma\frac{n(e^{-tR/a}a^2/b)^{n-1}}{r^{n+1}}P_n. \quad .\ (14)$$

Comparing the first of these results with (9) and the second with (10), we have the following conclusions:—

(i.) At time t after the generation of the magnet, the potential due to the induced currents is, at points inside the sphere, equivalent to that produced by a magnet of moment $-Me^{3tR/2a}$ at a distance from the centre of $e^{tR/a}b$.

(ii.) At points outside the sphere, the potential is equivalent to that of a magnet of moment $-Me^{-3tR/2a}a^3/b^3$ at a distance from the centre of $e^{-tR/a}a^2/b$. The positions P, Q occupied by these images at time t are always inverse points.

It is easily seen that the intensities of the images at P and Q are to each other and to that of the inducing magnet as $OP^{\frac{3}{2}} : OQ^{\frac{3}{2}} : OA^{\frac{3}{2}}$, that is in the sesquiplicate ratio of their distances from the centre.

Images due to Generation of a Small Magnet inside a Spherical Sheet.

9. If $b<a$ or the magnet is generated within the sphere, we must expand the potential in the second form (10) in order to substitute in the surface-conditions. When this is done, it will be found that

$$A_n = Me^{-(2n+1)tR/2a}\frac{b^{n-1}}{a^n}$$

and the conclusions finally arrived at are as follows :—

(i.) If a small magnet of moment M be instantaneously generated at a point B inside the sphere at a distance b from the centre, the axis of the magnet being radial; then the image representing the magnetic potential of the induced currents outside the sphere at time t is a magnet of moment $-Me^{-3tR/2a}$ at a point Q such that $OQ = be^{-tR/a}$.

(ii.) At points inside the cylinder the potential is that which would be produced by a magnet of moment $-Me^{3tR/2a}a^3/b^3$ at a distance from the centre of $e^{tR/a}a^2/b$.

As before, the intensities of the several magnets vary as the $\frac{3}{2}$ power of their distance from the centre.

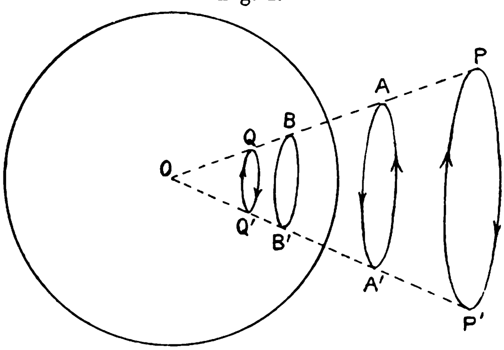
Images due to Generation of Circular Currents coaxial with a Spherical Sheet, &c.

10. A circular current whose axis passes through the centre of the sphere is equivalent to a magnetic shell forming a spherical cap on the surface of a concentric sphere, this cap being bounded by the current. It is thus evident that the images due to the sudden generation of such a current are also circular currents, which move in the manner described in the preceding sections. There is no need to restrict the results to

circular currents as the same arguments are equally applicable when the inducing current flows in any closed curve traced on a spherical surface concentric with the sheet, the images in this case being currents in similar curves. It remains to state the relation between the intensities of the currents in the images and that of the inducing current.

(i.) Let AA′ (fig. 4) be the wire in which an inducing

Fig. 4.



current of intensity I is suddenly generated *outside* the sheet. Then at time t the image for inside points will be a current in PP′ and for outside points a current in QQ′, where $OP = e^{tR/a}OA$ and $OQ = e^{-tR/a}a^2/OA$. Suppose the currents replaced by their equivalent magnetic shells on spherical surfaces centred at O, and let corresponding elements of these shells be taken. The magnetic moments of the elements of PP′, QQ′, and AA′ are respectively as $OP^{\frac{3}{2}} : OQ^{\frac{3}{2}} : OA^{\frac{3}{2}}$; and the areas of these elements are as $OP^2 : OQ^2 : OA^2$; hence the strengths of the shells are as $OP^{-\frac{1}{2}} : OQ^{-\frac{1}{2}} : OA^{-\frac{1}{2}}$, and the current-intensities are in the same ratio. Thus finally, if I′ and I″ be the intensities of the image-currents at time t in PP′ and QQ′, we have

$$I' = -\left(\frac{OP}{OA}\right)^{-\frac{1}{2}} I = -e^{-tR/2a}\, I,$$

$$I'' = -\left(\frac{OQ}{OA}\right)^{-\frac{1}{2}} I = -e^{tR/2a}\,\frac{b}{a}\, I.$$

It will be noticed that I″ increases as its circuit QQ′ approaches the centre, while I′ decreases as its circuit PP′ recedes to infinity.

(ii.) Next taking the inducing current I to be suddenly generated in BB′ *inside* the sphere, and taking $OB=b$, $OQ=e^{-tR/a}b$, and $OP=e^{tR/a}a^2/b$, we find that at time t, the image representing the potential of the induced currents at points outside the sphere is a current in QQ′ of intensity

$$-e^{tR/2a}\,I,$$

and at points inside the sphere a current in PP′ of intensity

$$-e^{-tR/2a}\frac{b}{a}\,I.$$

Images due to Generation of Magnetic Pole outside a Spherical Sheet.

11. A magnetic pole m generated at time 0 outside a sphere at the point $r=b$ where $b>a$ might be regarded as equivalent to a line of elements uniformly magnetized in the direction of the radius-vector extending along that radius from $r=b$ to infinity; and similarly a positive pole m inside the sphere at $r=b$ when $b<a$ together with a negative pole $-m$ at the centre might be regarded as equivalent to a line of elements uniformly magnetized in the line joining these points. In this way the images might be obtained by integration. Practically as short a way of obtaining the result as any is by utilizing the work of §§ 8, 9, taking into account the differences introduced into the coefficients of the various harmonics when for the inducing potential of a magnet there is substituted that of a pole.

When the pole is outside the sphere the inducing potential after its generation becomes

$$\Omega_0=m\Sigma\frac{r^n}{b^{n+1}}\,P_n\quad(r<b). \quad . \quad . \quad . \quad . \quad (15)$$

The coefficient of P_n is $-b/(n+1)$ of the corresponding coefficient for a magnet of moment m.

(i.) Hence we find for the potentials of the induced currents at points inside the sphere at time t

$$\Omega'=-m\Sigma e^{-(2n+1)tR/2a}\frac{r^n}{b^{n+1}}P_n$$

$$=-me^{tR/2a}\Sigma\frac{r^n}{(be^{tR/a})^{n+1}}\,P_n. \quad . \quad . \quad . \quad (16)$$

This is the potential of a pole of intensity $-me^{tR/2a}$ at a distance from the centre of $be^{tR/a}$.

(ii.) For points inside the sphere the corresponding potential is

$$\Omega''=m\Sigma\frac{n}{n+1}e^{-(2n+1)tR/2a}\frac{a^{2n+1}}{b^{n+1}r^{n+1}}P_n$$

$$=m\frac{a}{b}e^{-tR/2a}\Sigma\frac{(e^{-tR/a}a^2/b)^n}{r^{n+1}}$$

$$=m\frac{a}{b}e^{-tR/2a}\Sigma\frac{n}{n+1}\frac{(e^{tR/a}a^2/b)^n}{r^{n+1}}P_n.$$

Let $\rho=e^{-tR/a}a^2/b$. Then

$$\Sigma\frac{n}{n+1}\frac{\rho^n}{r^{n+1}}P_n=\Sigma\frac{\rho^n}{r^{n+1}}P_n-\frac{1}{\rho}\Sigma\frac{\rho^{n+1}}{(n+1)r^{n+1}}P_n$$

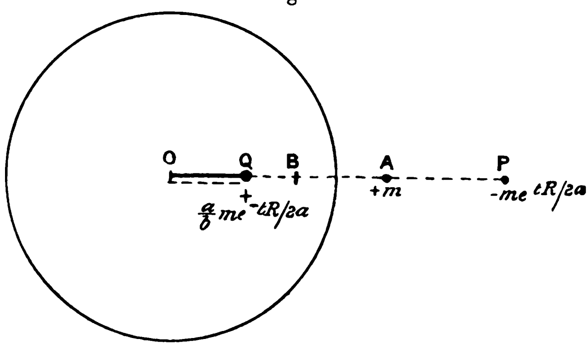
$$=\Sigma\frac{\rho^n}{r^{n+1}}P_n-\frac{1}{\rho}\int_0^\rho\Sigma\frac{\rho'^n}{r^{n+1}}P_n\,d\rho'$$

$$=\frac{1}{\sqrt{(r^2-2r\rho\cos\theta+\rho^2)}}-\frac{1}{\rho}\int_0^\rho\frac{d\rho'}{\sqrt{(r^2-2r\rho'\cos\theta+\rho'^2)}}$$

$$\therefore\ \Omega''=\frac{me^{-tR/2a}a/b}{\sqrt{(r^2-2r\rho\cos\theta+\rho^2)}}-\frac{m}{a}e^{tR/2a}\int_0^\rho\frac{d\rho'}{\sqrt{(r^2-2r\rho'\cos\theta+\rho'^2)}}. \quad . \ . \ . \ (17)$$

The first part represents the potential of a pole of strength $me^{-tR/2a}a/b$ at a distance ρ from the centre, *i.e.* at the point Q of the previous sections, and the second represents the potential of a line-distribution of magnetism of uniform linear

Fig. 5.



density $-me^{tR/2a}/a$ extending from the centre to Q; this being

equal to the strength of the pole divided by ρ or OQ, it follows that the total quantity of magnetism on this line OQ is equal and opposite to that in the pole Q (fig. 5). This distribution of images is related to the image on the opposite side in the same manner as the hydrodynamic image of a source in a sphere is related to the source, except in the matter of algebraic sign. The reason for the resemblance is that the two potentials here have to make $d\Omega'/dr = d\Omega''/dr$ at the surface, while in the hydrodynamic problem the portions of the normal velocity components $d\phi/dr$ due to the source and its image have to cancel one another.

Images due to Generation of Magnetic Poles inside a Spherical Sheet.

12. If a pole m be generated inside a sphere at the point $r=b$ where $b<a$, together with an equal and opposite pole $-m$ at the centre, the inducing potential takes the form

$$\Omega_0 = m\Sigma_{n=1}^{n=\infty} \frac{b^n}{r^{n+1}} \mathrm{P}_n. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(i.) The induced potential outside the sphere takes the form

$$\Omega'' = -me^{-t\mathrm{R}/2a}\Sigma_{n=1}^{n=\infty} \frac{(be^{-t\mathrm{R}/a})^n}{r^{n+1}} \mathrm{P}_n. \quad . \quad . \quad . \quad . \quad (19)$$

This is the potential of a pole of intensity $-me^{-t\mathrm{R}/2a}$ at a distance from the centre of $be^{-t\mathrm{R}/a}$, together with an equal and opposite pole at the centre.

(ii.) The induced potential inside is

$$\Omega' = m\Sigma \frac{n+1}{n} e^{-(2n+1)t\mathrm{R}/2a} \frac{r^n b^n}{a^{2n+1}} \mathrm{P}_n.$$

If $\rho = e^{t\mathrm{R}/a}a^2/b$, this becomes

$$\Omega' = m\frac{a}{b} e^{t\mathrm{R}/2a}\Sigma \frac{n+1}{n} \frac{r^n}{\rho^{n+1}} \mathrm{P}_n$$

$$= m\frac{a}{b} e^{t\mathrm{R}/2a} \left\{ \Sigma \frac{r^n}{\rho^{n+1}} \mathrm{P}_n - \frac{1}{\rho} \int_\rho^\infty \Sigma \frac{r^n}{\rho'^{n+1}} d\rho' \right\}. \quad . \quad (20)$$

The images thus consist of a positive pole of strength $me^{t\mathrm{R}/2a}a/b$ at a distance from the centre ρ equal to $e^{t\mathrm{R}/a}a^2/b$ together with a line-distribution of magnetism of linear

density $-me^{-tR/2a}/a$ extending from distance ρ to infinity. This linear density is equal to the strength of the pole-image divided by its distance ρ.

Other Images in a Spherical Sheet.

13. When the inducing potential is due to a small *magnet whose axis does not pass through the centre* of the sphere we could of course find the images by considering the positive and negative poles separately. If the axis of the magnet makes an angle α with the radius-vector the simplest plan would be to replace it by two magnets of moment M $\cos\alpha$ and M $\sin\alpha$ along and perpendicular to the radius.

14. The images due to *a pole of constant strength moving along the radius-vector* and initially at rest will be trails of magnets which can be worked out by employing the simpler images of §§ 10 and 11. If, initially, the pole were *suddenly generated*, we should have to introduce the more complicated images of §§ 11 and 12, but this may be avoided by considering the problem in which the pole is supposed to have remained at rest from an infinite time past (or, at any rate, so long that all currents in the sheet have died out), and investigating only the effects due to its motion. As usual, any small displacement may be represented by putting down a magnet of which one pole cancels the moving pole in its old position, while the other is identical with the moving pole in its new position. If the pole moves in a direction other than radial, the more complicated images of §§ 11 and 12 will of course be introduced.

General Conclusions.

1. The phenomena of two-dimensional induction in cylindrical sheets and of induction in spherical sheets due to a sudden disturbance in the magnetic field can be represented by moving trails of images which are generally not much more complicated than those obtained with a plane sheet.

2. The images in every case start from the source of disturbance and its inverse, and move away from the sheet radially with velocities varying directly as the distance, the constant of variation being such as to make the velocity at the surface of the sheet become equal to R, the corresponding

velocity for a plane sheet of the same thickness and conductivity. The image which initially coincides with the source of disturbance initially neutralizes its effect on the opposite side of the sheet.

3. Where the source of disturbance is a distribution of poles or an unclosed current *inside* the sheet it is necessary from physical considerations to place equal and opposite poles at the centre or to close the circuit by axial return currents, and this is also essential for the success of the image method.

4. The relations between the intensity of the source of disturbance and those of the images will readily be found to be in every case expressible in terms of their distances from the centre. The results expressed in this way may be tabulated as follows :—

Sheet.	Inducing disturbance.	Law of variation.
Cylinder.	Line of poles parallel to axis.	Constant.
,,	Current parallel to axis.	,,
,,	Line of magnets.	$\propto$ distance.
Sphere.	Magnet radially magnetized.	$\propto$ (distance)$^{\frac{3}{2}}$.
,,	Concentric magnetic shell. Current on concentric sphere.	$\propto$ (distance)$^{\frac{1}{2}}$.
,,	Magnetic pole (*with exception*).	$\propto$ (distance)$^{\frac{1}{2}}$.

5. The last-named exceptional case is that of a spherical sheet when the disturbance is due to the generation of a pole, and it is required to represent by images the potential of the currents on the same side as the disturbing pole, the images thus being on the opposite side of the sheet. Here, in addition to the single pole whose intensity and position are given by the above statements, the image consists of a line-distribution of magnetism of oppositt sign, extending from that pole along the radius away from the sheet, and having its line-density equal to the quantity of magnetism in the pole or image divided by its distance from the centre.

6. When the source of disturbance is in motion, the trails of images can be built up synthetically as in the case of a plane sheet. If the source of disturbance is rotating

uniformly about the centre of the cylinder or sphere, the curves described by the trails of images will be equiangular spirals.

Discussion.

Dr. S. P. Thompson said the method and the results obtained would find useful application in the solution of many allied problems.

XVI. *On the Properties of Liquid Mixtures.*—Part II. *By* R. A. Lehfeldt, *Professor of Physics at the E. London Technical College**.

In a previous article (Phil. Mag. (5) vol. xl. p. 398) an attempt was made to follow out the consequences of a certain thermodynamic relation between the composition of a liquid mixture, and that of the vapour in equilibrium with it, and the saturation pressure of the system. Experiments were there described on mixtures of benzene with ethyl and methyl acetates in which a small fraction of each mixture was distilled, as nearly as possible at a constant temperature, and the distillate analysed; these led to an empirical expression connecting the composition of liquid and vapour. Experiments were afterwards made to determine the vapour-pressure of similar mixtures by the dynamic method, but they led to unsatisfactory results owing to decomposition of the esters on prolonged boiling. I was able to resume the work in 1897, and then chose more stable compounds to work on, viz., benzene and toluene mixed with carbon tetrachloride, as types of normal organic compounds, and benzene and toluene mixed with ethyl alcohol, as type of a so-called "associated" liquid. The experiments to be described here were carried out at the Davy-Faraday laboratory, which the managers of the Royal Institution very obligingly placed at my service.

Two other papers on the same subject, which appeared about the same time as my earlier one, call for some notice.

* Read March 11, 1898.

These are by C. E. Linebarger (Journ. of Amer. Chem. Soc. vol. xvii.) and by M. Margules (*Wien. Ber.* vol. civ.).

Linebarger made measurements of the vapour-pressures of certain liquid mixtures, and of the way in which the two components shared the pressure between them. He did not attempt to relate his results to the deductions of thermodynamics, but merely to obtain empirical generalizations. The method he adopted is at first sight a very promising one, as it allows of the determination of both total and partial pressures in the same experiment; it consisted in drawing a measured volume of air through the liquid mixture, and analysing by combustion the vapour which the air carried away with it. The method has been applied successfully to find the vapour-pressure of aqueous solutions of low volatility, but it is not so suitable to the present case, as in Linebarger's experiments the vapour-pressure was sometimes as high as 300 mm., and the inaccuracies of it appear to increase out of proportion to the pressure to be measured. Linebarger tested his apparatus by preliminary measurements of pure liquids: finding for instance, for ethyl iodide 199 mm. against Regnault's 206, and for chloroform 290·1 against 301·1. This he calls a "most excellent correspondence." I tried the method before reading the account of his results, and found similar discrepancies of one or two per cent., so that it can evidently not be regarded as satisfactorily worked out as yet. It has also the disadvantage of being so slow that it is impossible to get numerous data. I have, therefore, preferred to revert to the better known methods, and nearly all the observations mentioned below were taken by the "dynamic" method.

Linebarger gives an empirical result which may be considered along with the observations contained in this paper: that strictly normal liquids, such as benzene and toluene, have in mixtures a partial pressure simply proportional to the molecular percentage of them present in the liquid. It is important, as on it Linebarger bases a rule for determining the molecular complexity of liquids. To this point we shall have to revert.

Margules's paper consists only of deductions from the previous experiments of others; but it includes a theorem given

in my paper (*loc. cit.*) as well as several others; and in an appendix mentions that all these results have been forestalled in the very systematic thermodynamical studies of Duhem—a fact which I also had overlooked. Margules's chief new contribution is, therefore, the empirical formula he proposes for the relation between composition of liquid and vapour, as will be mentioned below.

Measurements of Vapour-Pressure.

The measurements made form two distinct groups, those of vapour-pressure and those of composition of vapour, which were carried out separately, but on material from the same source, and prepared in identical manner. The materials were as follows:—

Benzene.—Kahlbaum's thiophene-free: its boiling-point was nearly constant, but it was fractionated twice from calcium chloride, and then stood over sodium. B.p. (corrected and reduced to 760 mm.) $=80^\circ{\cdot}0$. Vapour-pressure at $50^\circ=270{\cdot}9$ mm. Density $\frac{18}{4}=0{\cdot}8803$. μ_D at $18^\circ=1{\cdot}5024$ by spectrometer.

Toluene.—Baird and Tatlock's purest. Fractionated from calcium chloride and stood over sodium. B.p. (corr. and red.) $=110^\circ{\cdot}0$. Vapour-pressure at $50^\circ=93{\cdot}0$ mm. Density $\frac{18}{4}=0{\cdot}8667$. μ_D at $18^\circ=1{\cdot}4970$.

Carbon Tetrachloride.—Baird and Tatlock. Slightly yellow when obtained; on fractional distillation it became colourless, and on redistilling the best fraction nearly the whole came over within $0^\circ{\cdot}1$. B.p. $76^\circ{\cdot}6$ (corr. and red.). Vapour-pressure at $50^\circ=310{\cdot}2$ mm. Density $\frac{18}{4}=1{\cdot}5979$. μ_D at $18^\circ=1{\cdot}4618$.

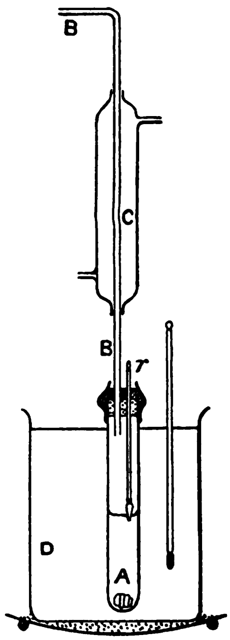
Ethyl Alcohol.—Baird and Tatlock's "absolute." Digested on a water-bath with lime and baryta till a yellow colour appeared; then distilled with special care to avoid moisture. B.p. (corr. and red.) $=78^\circ{\cdot}3$. Vapour-pressure at $50^\circ=219{\cdot}5$ mm. Density $\frac{18}{4}=0{\cdot}7929$. μ_D at $18^\circ=1{\cdot}3622$.

The density is not so low as it should be according to Mendeléef; but it is probable that if a purer alcohol had been

obtained it would have absorbed some moisture in the course of the inevitable manipulation of the mixtures. Distillation from sodium was also tried, but did not give any better results.

To measure the vapour-pressure of the mixtures, the dynamic method was adopted. The most interesting part of the apparatus is shown in the accompanying sketch (fig. 1); it consists of a boiling-tube A about 15 cm. × 3, closed at the top by a cork with two holes. Through one passed the thermometer—a short one with milk glass scale 40° to 60° in $\frac{1}{5}$; its bulb was surrounded with a little cotton-wool, and an end of the wool hung down into the liquid, so that the bulb was always moist; if that is the case, and the stem does not touch the side of the boiling-tube, the readings are very trustworthy. Through the cork there passed also a tube B, about 9 mm. wide, surrounded by a short condenser, bent at right angles above, and leading to (1) a mercury manometer, (2) a T-piece, of which the vertical limb passed into a Winchester, serving as a reservoir of air, and containing a little strong sulphuric acid; and the further limb a glass tap. The necks of the boiling-tube and of the winchester were surrounded by a short piece of wide rubber tubing each, and the corks drowned in mercury; the only other joints in the apparatus were the two rubber tubing joints of the manometer and the glass tap; it could be made absolutely air-tight without trouble. On the far side of the glass tap was placed a T-piece leading to a hand air-pump on one side, and on the other into the atmosphere, through a long capillary glass tube which served to reduce the flow of air; the pressure in the apparatus could thus be adjusted with any degree of nicety. The boiling-tube

Fig. 1.



contained about 10 to 15 grams of liquid mixture, previously made up by weighing, and a piece of pumice-stone weighted with copper wire, to make the ebullition steady; although the thermometer was surrounded by vapour, its readings were never constant unless the liquid was boiling freely. Heat was applied by means of a water-bath, consisting of a two-litre beaker, heated over a sand-tray, with a small flame; this was provided with a stirrer and thermometer, and kept three or four degrees above the temperature at which the mixture boiled.

An experiment consisted in weighing out a mixture, taking its refractive index by the Pulfrich refractometer, placing in the boiling-tube, and after adjusting temperature and pressures, taking eight observations at temperatures rising from 45° to 55°, and then falling to 45° again, and taking the refractive index of the residue: the refractive index was throughout used as a means of analysis (as in the previous paper, *q. v.*), and it showed that no appreciable change of composition took place during the experiments.

The instruments used were the following:—(i.) A cathetometer (by Fuess) to read the manobarometer with. Its scale was taken as correct; and since readings were only taken to one-tenth of a millimetre, no difficulty was encountered. Pressures were in all cases corrected for temperature of the mercury, and reduced to sea-level in lat. 45°.

(ii.) Five thermometers, of which the most important were "A" of range 40° to 60° in $\frac{1}{5}$, used for the boiling-tube, and "B" of range $-5°$ to $+35°$ in $\frac{1}{5}$, used for the refractometer, both by C. E. Müller. Their errors, which were very small, were determined by comparison with the standards of the Reichsanstalt, nos. 7346 and 7347, belonging to the Davy-Faraday laboratory.

(iii.) Pulfrich refractometer of the old pattern (by Max Wolz). Measurements of the refractive indices of alcohol (1·36), carbon tetrachloride (1·46), and benzene (1·50) were taken by it and a small Schmidt and Haensch spectrometer. The values given by Pulfrich's table were thus found to be too low by 0·00620 for alcohol, 0·00613 for carbon tetrachloride, and 0 00615 for benzene. Accordingly 0·0062 was

added to the number from the table in each case. Measurements of the refractive index of the mixtures were always taken as near to 18° as possible (though unfortunately in the summer the room rose frequently to 21°): the temperature-coefficients were interpolated from the known coefficients of the pure substances (this involves very little error as the coefficients only vary from 0·00040 for alcohol to 0·00064 for benzene), and the readings reduced to the standard temperature.

An accuracy of the order 1 in 1000 was aimed at in the various measurements; but of course the final results are hardly reliable to that extent. To take a typical case, the saturation-pressure to be measured might be about 250 mm.: there is of course no difficulty whatever in reading the height of the mercury-column to one thousandth of that ($\frac{1}{4}$ mm.). The corresponding temperature needs to be known to about $\frac{1}{40}$ degree, since 1° would make about 10 mm. difference in pressure. The thermometer was read to $\frac{1}{50}$ by eye with certainty; and when the apparatus was in good working order, it kept constant to that extent. Again, to change the vapour-pressure by $\frac{1}{1000}$ (temperature constant) would need a change usually as great as $\frac{1}{500}$ or more in the composition; this would correspond to about 2′ of arc on the refractometer—a quantity very easily observed. The composition of the mixture might alter by escape of the vapour past the condenser; this would be discovered by a comparison of the refractive index before and after the boiling—as a matter of fact, the change rarely exceeded 2′: further, it would necessarily alter as some of the substance was present in the form of vapour, but the weight of vapour in the apparatus could not have exceeded $\frac{1}{250}$ of the weight of liquid, and as the composition of the vapour is never 30 per cent. different from that of the liquid, the evaporation of that quantity could not alter the composition of the remainder more than about $\frac{1}{800}$. The degree of accuracy thus anticipated appears to have been realized except in a few unfavourable cases, as will be seen from the following observation, which is given as a specimen, and to show the mode of reduction adopted.

Specimen Observation.

Weighing-bottle	11·3608 grms.		
+ toluene . .	17·1159 „		52·40 per cent. of CCl_4.
+ CCl_4 . .	23·4510 „		

Refractometer before 40° 16½′ at 21°·1. Hence μ_{18}=1·48817.
„ after 40° 20½′ at 21°·8. „ =1·48809.

Height of		Pressure.	Obs. temp.	Corr. temp.	Log pressure.	Log p_{50}.
Barom.	Gauge.					
779·1	614·5	164·6	48°·14	48°·20	2·2164	2·2470
(779·4)	596·8	182·6	50·84	50·92	·2615	59
(779·6)	580·5	199·1	53·18	53·31	·2991	27
779·9	561·3	218·6	55·60	55·79	·3397	12
(779·6)	587·4	192·2	52·24	52·33	·2838	42
(779·4)	608·0	171·4	49·38	49·45	·2340	32
(779·2)	625·3	153·9	46·64	46·69	·1872	36
779·1	641·4	137·7	43·90	43·93	·1389	22
					Mean=	2·2437

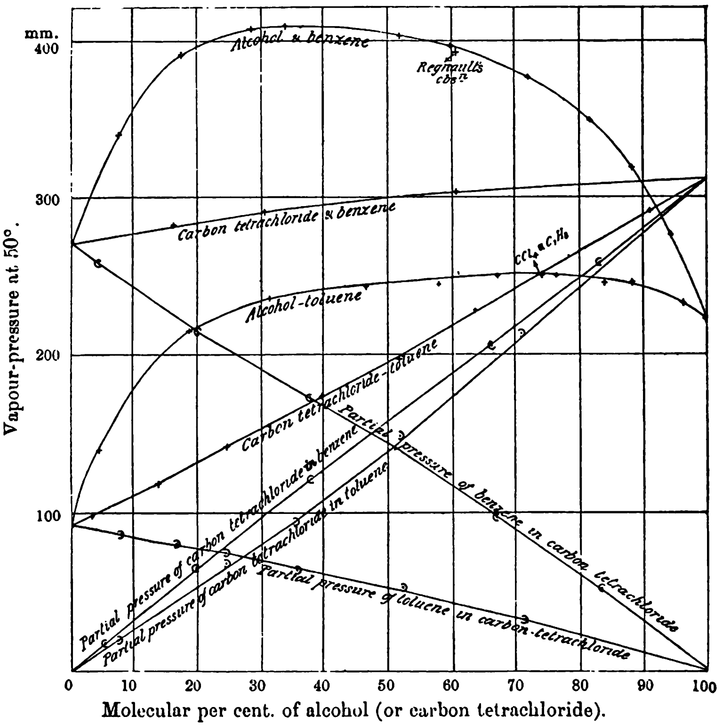
From curve $\frac{d(\log p)}{dt}$=0·01702.

The observed temperatures and logarithms of the pressure were, in practice, plotted on a curve (which was approximately a straight line), and the values of $\log p_{50}$ and $d(\log p)/dt$ read off. Here the values of $\log p_{50}$ have been calculated from each observation and the assumed temperature-coefficient, in order to show the degree of concordance obtained. The probable error of $\log p_{50}$=0·0005, whence p_{50}=175·3±0·1. This reduced to 0° and sea-level in lat. 45° becomes p_{50}=**174·8** mm. The results are shown in the accompanying diagram (fig. 2) and in the tables below. I have only been able to find one earlier observation on the same mixtures: from Regnault's experiments * on mixtures of alcohol and benzene at higher and lower temperatures one finds by interpolation p_{50}=391 mm. for a mixture in equal volumes of the liquids, whilst the value from my curve is 393 mm. It is satisfactory to find this confirmation, slight as it is, because the alcohol mixtures are

* Regnault, *Mém. de l'Acad.* xxvi.

the least certain, on account of the inevitable presence of traces of water.

Fig. 2.



The difference between the alcohol curves and those of the normal liquids is most marked; for while mixtures of carbon tetrachloride and benzene, or carbon tetrachloride and toluene, have a vapour-pressure never very different to that which would be obtained by interpolation from the pressures of the pure substances, both mixtures containing alcohol show a maximum—a very flat one. In the case of alcohol and benzene it occurs at about 40 per cent. (molecular) of alcohol, and is 406 mm., being $406/(271+220)=83$ per cent. of the sum of the pressures of the pure substances; while for alcohol and toluene it is at about 74 per cent. of alcohol, and has the value 249 mm., which is $249/(93+220)=80$ per cent. of the sum of the pressures.

With regard to the variation of pressure with temperature, we may, as already remarked, regard $\log p$ as linear in t over a few degrees. But the value of $\partial(\log p)/\partial t$ for the pure substances varies somewhat; this is partly due to the fact that 50° is a different fraction of the critical temperature in each case; it diminishes as the "reduced" temperature rises, and consequently has a higher value for toluene at 50° than for benzene or carbon tetrachloride at the same temperature. Alcohol, though its critical point is the lowest of the four, has the highest value of the coefficient—a marked instance of its exceptional behaviour. The numbers are:—

	$\partial(\log p)/\partial t$.
Toluene	0·0189
Benzene	0·0164
Carbon tetrachloride	0·0156
Alcohol	0·0209

The coefficients found for the mixtures are necessarily (on account of the small range of temperature) less reliable than the vapour-pressures themselves: they are in all cases nearly those that would be calculated by interpolation from the coefficients of the pure substances, even when the vapour-pressure curve shows a maximum, *i. e.* they are additive. The differences from the additive values are not greater than the probable errors of experiment. I have included the values of the coefficients in the tables, but lay no stress on them.

Alcohol and Benzene: Vapour-pressures at 50°.

Per cent. alcohol.	Mol. per cent. alcohol.	Saturation-press. p_{50}.	$\frac{d(\log p)}{dt}$.
0	0	270·9	0·0164
4·32	7·11	339·8	
11·11	17·49	389·9	0·0172
18·87	28·28	404·3	179
23·20	33·87	406·4	179
38·81	51·80	400·3	178
46·75	59·80	394·0	188
60·33	72·05	375·3	182
72·37	81·61	345·8	184
81·32	88·07	318·8	175
90·61	94·24	274·3	193
100	100	219·5	209

Alcohol and Toluene.

Per cent. alcohol.	Mol. per cent. alcohol.	Saturation-press. p_{50}.	$\frac{d(\log p)}{dt}$.
0	0	93·0	0·0189
2·14	4·20	141·2	
9·74	17·75	214·8	178
18·26	30·89	233·1	193
29·98	46·13	242·1	193
40·50	57·65	244·2	195
50·15	66·80	249·2	
61·65	76·27	248·2	
71·95	83·69	244·4	
78·60	88·02	243·0	
92·29	95·99	230·9	
100	100	219·5	209

In some of the above cases the temperature-coefficient was not found from experiment, but p_{50} derived from the additive value of it; therefore no number is recorded in the last column.

Carbon Tetrachloride and Benzene.

Per cent. CCl_4.	Mol. per cent. CCl_4.	p_{50}.	$\frac{d(\log p)}{dt}$.
0	0	270·9	0·0164
26·97	15·78	281·0	162
46·32	30·44	290·0	163
75·75	61·31	302·3	161
100	100	310·2	156

These two liquids have so nearly the same vapour-pressure, and the curve between them is so nearly straight, that three observations were considered enough.

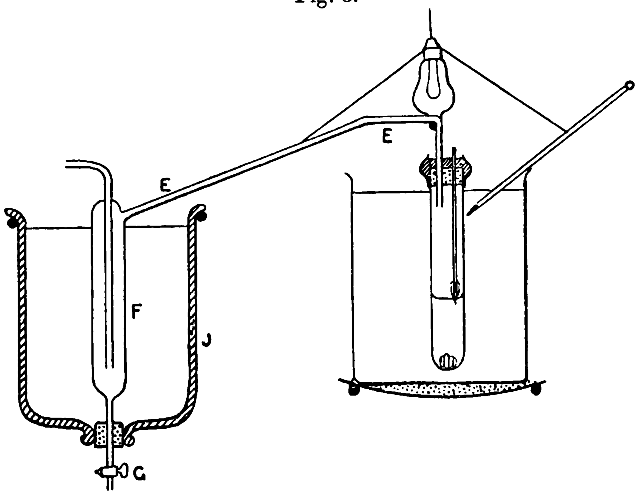
Carbon Tetrachloride and Toluene.

Per cent. CCl_4.	Mol. per cent. CCl_4.	p_{50}.	$\frac{d(\log p)}{dt}$.
0	0	93·0	0·0189
6·52	3·99	99·0	183
20·75	13·51	117·9	174
34·61	24·01	140·8	173
52·40	39·70	174·8	170
63·93	51·42	197·7	167
74·47	63·51	226·5	169
82·64	73·98	248·5	163
94·44	90·99	288·8	156
100	100	310·2	156

Composition of the Vapour over Liquid-Mixtures.

The method used was essentially the same as that described in the previous paper, to distil a little of a mixture and analyse the distillate. The apparatus was, however, arranged so that the distillate could be drawn off by a tap as required; it is shown in fig. 3. It consisted of a boiling-tube, fitted

Fig. 3.



and heated as before; but with the delivery-tube E E bent round to a condenser F. The condenser was provided with a tap G for drawing off the distillate, and a tube which connected to the pressure-gauge, reservoir, and air-pump arranged as before. The condenser was designed so as to permit of the use of a freezing-mixture in the bell-jar J by which it was surrounded; a good many experiments showed, however, that it made no difference to the result whether ice and salt or only cold water was used in J; so water was used for greater convenience. To make the apparatus air-tight, a crucible full of mercury was placed round G, since obviously it was not permissible to lubricate that tap. It is essential to the success of the experiment that no back-condensation should occur, but that the vapour should be collected exactly in the condition it is produced; so the delivery-tube before

the bend must be at least 50°, if the evaporation is taking place at that temperature. This condition was secured by a very simple device: an incandescent lamp with the ordinary conical shade was lowered as close as possible over the water-bath, and a cloth hung round the whole; the electrical heating was then sufficient to keep the top of the apparatus at least as hot as the small flame kept the bottom, and at the same time provided an excellent light to read the thermometers by. About 30 c.c. of mixture was placed in the tube; the temperature of the bath adjusted to between 51° and 52°, and then the pressure lowered till the liquid boiled tranquilly at as near as possible 50°; as the distillation proceeded, the thermometer of course tended to rise; this was corrected by an occasional stroke of the pump as required. Three lots of about 1 c.c. each were usually distilled, and examined separately by the Pulfrich refractometer; from 5 to 10 minutes being required to distil 1 c.c.

In order to interpret the results it was first needful to construct tables of the refractive index of the mixtures used; the refractive index is far from additive in any of the four cases, and cannot be satisfactorily expressed by the parabolic formula used in Part I. for mixtures of benzene with methyl and ethyl acetates; it is therefore worth while to record the results by a short extract from the tables used, which were themselves derived from a smoothed curve based on experiment.

Refractive Indices (at 18° for sodium light).

Per cent. of alcohol (or CCl_4).	Alcohol-benzene.	Alcohol-toluene.	Carbon tetrachloride-benzene.	Carbon tetrachloride-toluene.
0	1·5024	1·4970	1·5024	1·4970
10	1·4869	1·4823	1·5008	1·4956
20	1·4716	1·4680	1·4985	1·4937
30	1·4568	1·4539	1·4958	1·4914
40	1·4425	1·4401	1·4929	1·4889
50	1·4283	1·4265	1·4894	1·4860
60	1·4146	1·4131	1·4853	1·4827
70	1·4011	1·4000	1·4807	1·4787
80	1·3878	1·3873	1·4755	1·4742
90	1·3749	1·3747	1·4692	1·4685
100	1·3622	1·3622	1·4618	1·4618

The numbers recorded in an experiment on the composition of the vapour were (1) the refractometer reading, with the corresponding temperature for the liquid before distillation; (2) that for each of the distillates; (3) that of the residue. From them were calculated the refractive indices, reduced to 18°, and then, from the tables quoted above, the percentage composition of the liquid (mean between the reading before and after), and of the vapour (mean of the three distillates). The results are shown in the following tables :—

Composition of Vapours.

A, B = molecular weights of the two components.
z = fractional composition of liquid.
y = " " vapour.
ζ = molecular fractional composition of liquid.
η = " " " vapour.
q = ratio of masses of substances A and B present in liquid.
t = " " " " vapour.
χ = ratio of number of molecules of A and B present in liquid.
s = " " " " vapour.
p = total vapour-pressure.
p_A = partial pressure due to A.
p_B = " " B.

Alcohol–Benzene.

A (Alcohol) = 45·70. B (Benzene) = 77·46.

z.	y.	ζ.	η.	log q.	log t.	p (from curve).	p_A (from p & η).	p_B (from p & η).
0·05·4	0·18·7	0·08·8	0·28·1	$\bar{2}$·7566	$\bar{1}$·3617	350·4	98·5	251·9
0·07·5	0·21·9	0·12·1	0·32·2	$\bar{2}$·9090	$\bar{1}$·4478	369·0	118·8	250·2
0·13·9	0·24·7	0·21·5	0·35·7	$\bar{1}$·2079	$\bar{1}$·5159	397·0	141·6	255·3
0·24·5	0·27·5	0·35·5	0·39·1	$\bar{1}$·5112	$\bar{1}$·5790	406·0	158·7	247·2
0·32·0	0·29·7	0·44·4	0·41·7	$\bar{1}$·6727	$\bar{1}$·6258	404·4	168·6	235·8
0·43·0	0·32·6	0·56·1	0·45·1	$\bar{1}$·8776	$\bar{1}$·6846	397·6	179·3	218·3
0·58·0	0·38·3	0·69·7	0·51·3	0·1326	$\bar{1}$·7929	378·4	194·1	184·3
0·82·1	0·54·3	0·88·6	0·66·8	0·6615	0·0748	315·0	210·4	104·6

Alcohol–Toluene.

A = 45·70. B (Toluene) = 91·37.

z.	y.	ζ.	η.	log q.	log t.	p.	p_A.	p_B.
0·07·4	0·41·9	0·13·8	0·59·1	$\bar{2}$·9025	$\bar{1}$·8581	119·5	117·9	81·6
·20·0	·49·1	·33·4	·65·9	$\bar{1}$·3989	$\bar{1}$·9843	235·0	154·8	80·2
·28·0	·51·6	·43·7	·68·1	$\bar{1}$·5898	0·0278	241·0	164·1	76·9
·36·3	·54·4	·53·3	·70·5	$\bar{1}$·7558	0·0766	245·0	172·8	72·2
·46·4	·55·4	·63·4	·71·3	$\bar{1}$·9374	0·0941	247·0	176·1	70·9
·58·2	·58·6	·73·6	·73·9	0·1436	0·1508	249·0	184·0	65·0
·67·5	·62·1	·80·6	·76·6	0·3174	0·2146	246·5	188·8	57·7
·79·3	·72·3	·88·5	·83·9	0·5833	0·4166	241·5	202·6	38·9
·89·8	·82·1	·94·6	·90·2	0·9447	0·6615	233·5	210·6	22·9

Carbon tetrachloride–Benzene.

A (Carbon tetrachloride) = 152·70. B = 77·46.

z.	y.	ζ.	η.	log q.	log t.	p.	p_A.	p_B.
0·18·1	·10·1	·04·3	·05·4	$\bar{2}$·9450	$\bar{1}$·0508	273·6	14·7	258·9
·32·6	·37·4	·19·7	·23·3	$\bar{1}$·6846	$\bar{1}$·7763	280·0	65·2	214·8
·54·3	·58·0	·37·6	·41·2	0·0748	0·1402	294·0	121·1	172·9
·79·3	·80·5	·66·0	·67·7	0·5833	0·6157	303·4	205·3	98·1
·90·6	·90·9	·83·0	·83·5	0·9840	0·9995	306·5	255·8	50·7

Carbon tetrachloride–Toluene.

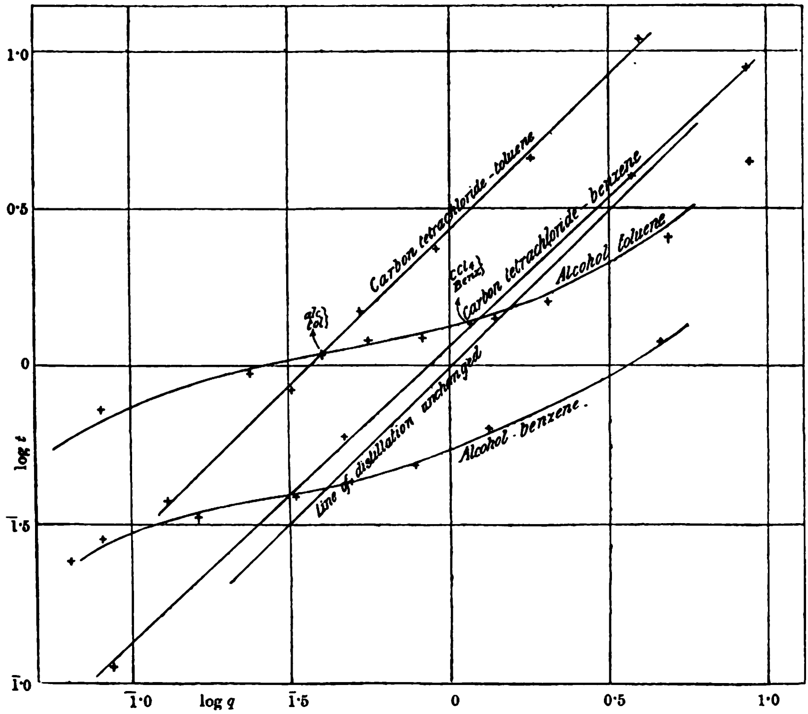
A = 152·70. B = 91·37.

z.	y.	ζ.	η.	log q.	log t.	p.	p_A.	p_B.
0·11·7	0·27·0	0·07·3	0·18·1	$\bar{1}$·1222	$\bar{1}$·5681	105·4	19·0	86·4
·24·6	·46·7	·16·3	·34·4	$\bar{1}$·5136	$\bar{1}$·9426	122·8	42·2	80·6
·34·5	·59·7	·24·0	·47·0	$\bar{1}$·7216	0·1706	140·0	65·8	74·2
·47·4	·71·2	·35·0	·59·7	$\bar{1}$·9548	0·3930	158·0	94·3	63·7
·64·1	·82·8	·51·7	·74·2	0·2519	0·6825	198·9	147·6	51·3
·80·2	·91·9	·70·8	·87·2	0·6076	1·0550	242·0	211·0	31·0

It is difficult to say what is the degree of accuracy of these results, but certainly less than the accuracy of the vapour-pressure measurements, chiefly because some of the condensed vapour clings to the condensing-tube, and it is not certain that what is collected quite fairly represents the whole.

These results are also expressed by the curves on fig. 4. Considering first, then, two pairs of normal liquids, we see that the relation $\log t = \log k + r \log q$, where k and r are

Fig. 4.



constants, satisfactorily represents the phenomena. That is the relation proposed in the previous paper for mixtures of benzene with methyl and ethyl acetates, and in all these cases it appears to be correct within the limits of error of experiment. But Margules has quite rightly pointed out (*loc. cit.* p. 1266) that when $r < 1$ (as it usually is) the equation leads to infinite values of $\partial p/\partial \zeta$ when $\zeta = 0$ or $\zeta = 1$, *i. e.*, the

vapour-pressure curve would start and end vertically. That deduction I had overlooked, and it is obvious from the curves on fig. 2 that the vapour-pressure curve does not behave in that way. The relation given cannot therefore hold throughout; still the experiments show it to be a remarkably good empirical rule for mixtures which do not contain a very small proportion of either component, extending in the case of carbon tetrachloride-benzene mixtures from 8 to 91 per cent.

The empirical linear relations are then :—

For carbon tetrachloride–benzene ... $\log t = 0{\cdot}065 + 0{\cdot}947 \log q$,
„ carbon tetrachloride–toluene ... $\log t = 0{\cdot}440 + 1{\cdot}0 \log q$.

These relations are intended as a solution of the differential equation (11) in the foregoing paper. Margules proposes a more complicated solution, which satisfies the condition that for dilute solutions the lowering of the vapour-pressure of the other component (the solvent) should be normal, *i.e.* according to van 't Hoff and Raoult's rule. It is doubtless possible to represent the experimental results that way by using a sufficient number of arbitrary constants; but I have not thought it worth while to do so, as it is complicated, and seems to go rather beyond what the experimental data justify.

With regard to the other mixtures, alcohol with benzene and toluene, the diagram shows the relations between $\log t$ and $\log q$ to be very far from linear. I have not attempted to find an equation to these curves, and will only make one or two remarks as to their character. They are not drawn directly from the experimental results on the composition of the distillate, because when the percentage of alcohol is very great or very small the observations lose considerably in accuracy; but on calculating the partial pressures, from the observed composition of the vapour, and the total pressure, smoothed curves could be drawn, the reliability of which is enhanced by the fact that their end points (vapour-pressure of the pure substances) are fixed. From these smoothed curves of partial pressure the compositions were calculated afresh, and it is the numbers so obtained that are represented by the curves on fig. 4. The crosses on that figure stand for the immediate results of observation, and it will be

noticed that they are regular enough in the middle parts of the curves but not at the ends.

At the maxima of vapour-pressure the composition of the liquid and the vapour must be the same. That is shown, consequently, by the intersection of the curves with the line $\log q = \log t$. The points of intersection form the most reliable measure of the position of the maxima. They give

	$\log q$.	z.	ζ.
Alcohol-benzene......	1·605	0·28·7	0·40·6
Alcohol-toluene......	0·170	0·59·7	0·74·7

It will be seen that these results agree well with the maxima shown in the vapour-pressure curves (fig. 2).

The differential equation referred to above is

$$\left(\frac{\mathrm{B}q}{\mathrm{B}q+\mathrm{A}} - \frac{s}{s+1}\right)\frac{1}{s}\frac{\partial s}{\partial q} + \frac{1}{p}\frac{\partial p}{\partial q} = 0,$$

or, as it may be written,

$$(\eta - \zeta)\frac{\partial \log s}{\partial \log q} = \frac{\zeta(1-\zeta)}{p}\frac{\partial p}{\partial \zeta}.$$

For some purposes it is more convenient to express it in terms of the partial pressures, when it assumes the more symmetrical form adopted by Margules (*loc. cit.*) :

$$\zeta\frac{\partial \log p_{\mathrm{A}}}{\partial \zeta} + (1-\zeta)\frac{\partial \log p_{\mathrm{B}}}{\partial \zeta} = 0.$$

Since, however, the relation between q and s (or between ζ and η) has not so far been expressed successfully by an equation, it is not possible to integrate the differential equation in order to compare it with experiment. This is true even for the mixtures of benzene and toluene with carbon tetrachloride, since the empirical equations, quoted above, do not extend through the whole range of q, and therefore the arbitrary constant in the integral equation cannot be determined. We are reduced therefore to the very rough process of comparing the differential equation itself with

experiment, by measuring the slope of the vapour-pressure curves at various points. The relation $\log t = \log k + r \log q$ or $\log s = \log Bk/A + r \log q$ gives $\partial(\log s)/\partial(\log q) = r$, and hence

$$\eta - \zeta = \frac{\zeta(1-\zeta)}{rp}\frac{\partial p}{\partial \zeta}.$$

The agreement shown between the two sides of this equation is very rough. Thus:

Carbon tetrachloride and Toluene.

ζ.	0·073	0·163	0·240	0·350	0·517	0·708
$\eta - \zeta$.............	0·105	0·181	0·230	0·247	0·225	0·164
$\frac{\zeta(1-\zeta)}{rp}\frac{\partial p}{\partial \zeta}$	0·123	0·233	0·276	0·311	0·278	0·198

Carbon tetrachloride and Benzene.

ζ.	0·043	0·197	0·376	0·660	0·830
$\eta - \zeta$.............	0·011	0·036	0·036	0·017	0·005
$\frac{\zeta(1-\zeta)}{rp}\frac{\partial p}{\partial \zeta}$	0·010	0·037	0·040	0·017	0·009

Whether the disagreement is due to the difficulty of treating the differential equation directly, to errors of experiment, or to the approximations in the theory, I am at present unable to say.

In the case of the alcohol mixtures Margules's form of the equation was adopted, and the differential coefficients measured from the smoothed curves. The results following show, it will be seen, the kind of agreement that might be expected in the middle of the range, but in the case of alcohol and benzene disagree totally at the extreme percentages.

Alcohol and Benzene.

ζ.	0·10	0·30	0·50	0·70	0·90
$\frac{\zeta}{p_A}\frac{\partial p_A}{\partial \zeta}$	0·67	0·19	0·27	0·34	0·40
$\frac{1-\zeta}{p_B}\frac{\partial p_B}{\partial \zeta}$	0·31	0·16	0·34	0·46	0·82

Alcohol and Toluene.

ζ.	0·10	0·30	0·50	0·70	0·90
$\frac{\zeta}{p_A}\frac{\partial p_A}{\partial \zeta}$	0·61	0·23	0·21	0·27	0·62
$\frac{1-\zeta}{p_B}\frac{\partial p_B}{\partial \zeta}$	0·54	0·22	0·25	0·27	0·70

It has already been mentioned that Linebarger states the conclusion that the partial pressure of benzene and toluene in mixtures is simply proportional to the molecular percentage present; this conclusion appears to be only roughly true. According to the results expressed by the curves on fig. 2 no such simple relation holds; the partial pressures of these hydrocarbons being as far from linear as that of the carbon tetrachloride with which they are mixed.

Linebarger proceeds to apply his conclusion to determine the molecular weight of acetic acid, in mixtures of the acid with benzene and toluene.

For in any mixture the partial pressure of the hydrocarbon p_B divided by the pressure of the pure liquid π_B gives, according to this rule, the true molecular fraction of hydrocarbon present, say $1-\zeta'$, whilst $1-\zeta$ is that calculated from the formulæ C_6H_6 (or C_7H_8) and CH_3COOH, or

$$1-\zeta = p_B/\pi_B,$$

whence

$$\chi' = \zeta'/(1-\zeta') = (\pi_B - p_B)/p_B,$$

where χ' is the true ratio of the number of molecules of acetic

acid to molecules of hydrocarbon. If this be divided into the ratio χ calculated from the formulæ, the quotient will give the degree of aggregation of the molecules of acetic acid in the solution. In this way Linebarger found molecular weights which steadily increased with the concentration of the acid, and by extrapolation found 240 for the molecular weight of the pure liquid at 35°. My results for alcohol do not, however, at all confirm the accuracy of the method. On drawing smooth curves of partial pressure from the observations given in the tables above, and treating them in the way just indicated, we get

Molecules of Alcohol.	Aggregation of the Alcohol Molecules in	
	Benzene.	Toluene.
10	6·0	2·0
30	5·7	2·6
50	5·3	4·2
70	5·0	5·7
90	4·9	6·0

It will be seen that the two series disagree altogether, and that neither leads to any reasonable value for the aggregation of pure alcohol. I can only conclude that the partial pressure of the hydrocarbon vapour is not necessarily linear in mixtures, and that therefore the rule proposed by Linebarger for determining the molecular weight of liquids is not correct.

Further theoretical conclusions would not, I think, be justified at the present time on account of the small amount of material accumulated by experiment.

Davy-Faraday Laboratory of the Royal Institution,
London, December 1897.

XVII. *On some Improvements in the Roberts-Austen Recording Pyrometer, with Notes on Thermo-Electric Pyrometry.* By
ALFRED STANSFIELD, *B.Sc., A.R.S.M., Royal College of Science, London* *.

THE method of recording pyrometry which involves the use of the thermo-junction has, as is well known, been devised by Prof. Roberts-Austen. He suggested that the author should undertake the investigations described in the following paper, and the experiments were conducted in the laboratory of the Royal Mint.

Introduction.

An excellent summary of the early work on pyrometry is given by Barus in his elaborate paper "On the Thermo-Electric Measurement of High Temperatures" †, published in 1889, so that it will only be necessary here to consider briefly the present condition of high-temperature measurement.

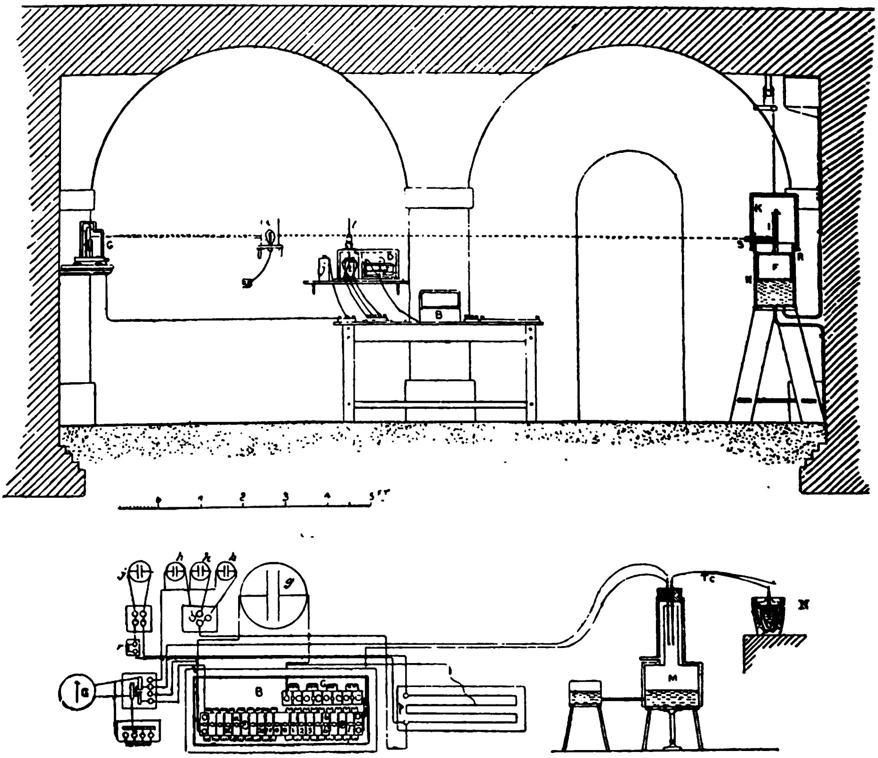
The ultimate standard of temperature is afforded by the air-thermometer; nearly all other instruments are referred to this either directly or indirectly. Apart from the air-thermometer, which is not well adapted for ordinary use, there are two systems of pyrometry suitable for exact measurements of temperature. One of these, which depends on the change produced in the electrical resistance of a coil of platinum wire by a change of temperature, was adapted to industrial use by Sir W. Siemens, and has been greatly improved and widely utilized by Callendar and Griffiths, and by Heycock and Neville. The other system utilizes the electromotive force produced in a circuit of two or more metals when one junction is heated; its early use is associated with the names of Becquerel (1826) and Tait (1873), while more recently it has been employed by H. Le Chatelier, Barus, and Roberts-Austen. The thermo-couples in general use consist of platinum and platinum alloyed with about 10 per cent. of either

* Read March 25, 1898.
† Barus, Bull. U.S. Geol. Survey, no. 54. Washington, 1889.

iridium or rhodium, the former alloy having been first used by Tait, the latter by Le Chatelier.

The platinum-resistance pyrometer, or platinum thermometer, as it is often called, appears to be capable of affording more accurate results than the thermo-couple especially up to about 600°. This mainly arises from the fact that the measurement of a resistance can be made with a greater

Fig. 1.—Arrangement of the Apparatus.



degree of accuracy than the measurement of the extremely small E.M.F. produced by the thermo-couple. Moreover, the whole of the coil of wire in the resistance-pyrometer is in the region the temperature of which is to be measured; hence the indications do not depend, as do those of the thermo-couple, on the absolute uniformity of composition of the

platinum wire employed. At higher temperatures, however, the resistance-pyrometer is more seriously affected than is the thermo-couple by the difficulty of securing good insulation, and by the difficulty of obtaining uniformity and constancy of the temperature to be measured; although the difficulty of insulation has been practically overcome by Callendar's use of mica supports. The resistance-pyrometer, therefore, although it may be capable of affording more accurate results than the thermo-couple, is much less convenient for use at high temperatures. The latter, consisting of two thin wires which may be inserted into a small protecting tube of fireclay or porcelain, can be used to measure the temperatures of very small enclosures, and its relatively small lag renders it suitable for obtaining autographic temperature records.

The present research was begun with the object of modifying the Roberts-Austen pyrometer, in order that it might be used to obtain large-scale temperature records having as high a degree of accuracy as possible. It was intended to study the degree of constancy which could be obtained in the indications of the thermo-couple; to make observations of the melting-points of several metals by means of a thermo-couple; and to calibrate the couple by means of the porcelain air-thermometer. The present paper describes the Roberts-Austen recording pyrometer, and the changes which have been made in view of the above objects, together with the precautions which should be observed in order to avoid errors in the use of the thermo-couple. It also contains an account of the results that have been obtained, and the bearing of these results on the theory of the thermo-couple.

The Roberts-Austen Recording Pyrometer.

The thermo-couple has been used in pyrometry in two ways:—(1) by measuring the E.M.F. of the heated couple by a null method, and (2) by observing the deflexion of a galvanometer to which it is connected.

In the recording pyrometer of Roberts-Austen * the latter method is adopted; the thermo-couple is connected to a galvanometer the deflexion of which is recorded on a moving

* Proc. Roy. Soc. 1891, vol. xlix. p. 347.

photographic plate. The resistance of the galvanometer is sufficiently large to render changes in the resistance of the thermo-couple unimportant, and as the instrument is calibrated directly by observations of known melting or boiling temperatures, it is not necessary to assume any definite relation between the current and the deflexion of the galvanometer. This form of pyrometer is extremely convenient and useful, and the photographic records taken by its aid of the cooling of an alloy of two or more metals frequently show small evolutions of heat which could not easily be detected by direct observation. These evolutions of heat, even when extremely small, are of great importance in studying the constitution of alloys. A few records obtained in this way are given in fig. 2, which is a reproduction of a photographic plate on which several records have been taken in succession. Three of these, B, C, and D, represent the cooling of alloys of copper and tin * containing respectively 50, 55, and 45 per cent. of copper.

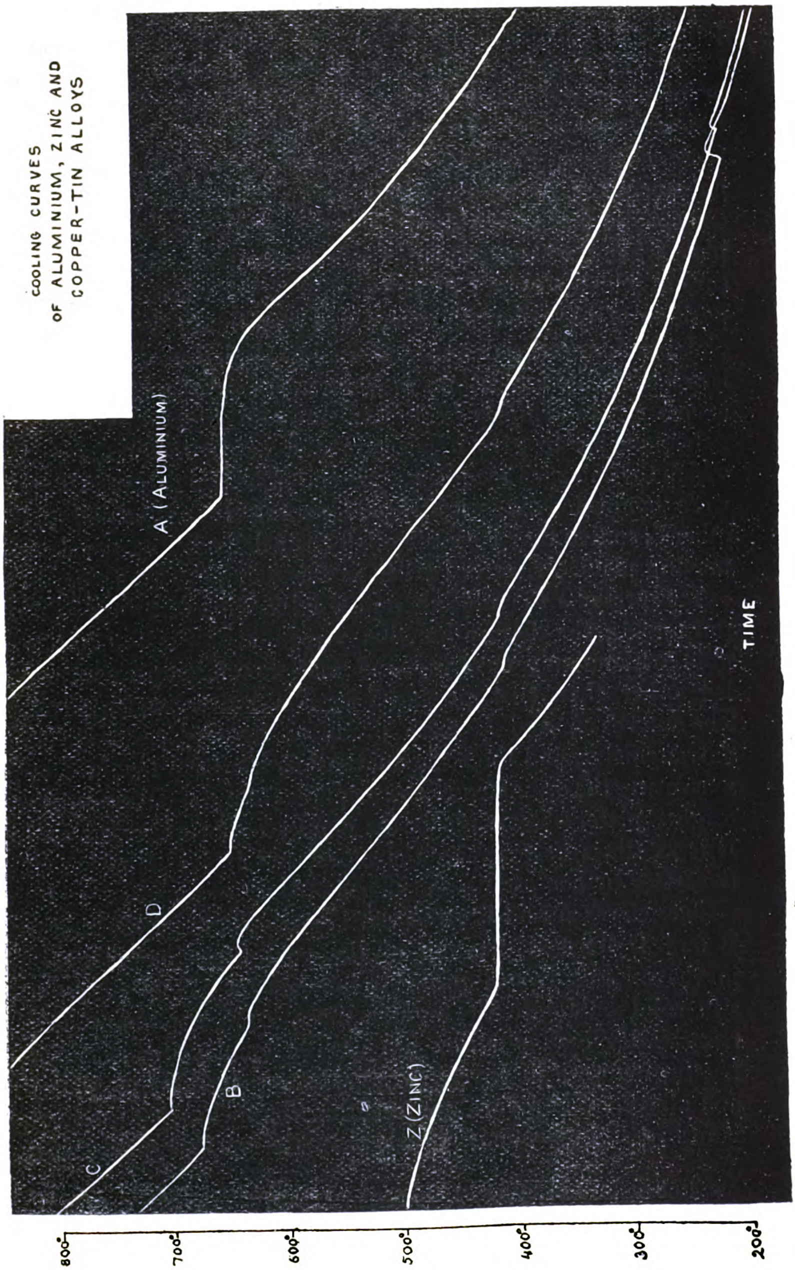
In these alloys, most of the halts in the cooling-curves represent the solidification of eutectic alloys. The curves A and Z represent the cooling and solidification of aluminium and of zinc, and the known melting-points of these metals have enabled a temperature-scale to be constructed for use with the other records.

In this form of recording pyrometer the accuracy of the record depends on the constancy of the galvanometer, as well as on that of the thermo-couple itself, so that another element of uncertainty is introduced into the results. In addition to this, the deflexion of the galvanometer for the highest temperature to be measured must be limited to the width of the photographic plate employed if the zero of the scale is to be recorded on the same plate. In measuring high temperatures, therefore, the sensibility of the galvanometer must be small, and consequently the record will be on a small scale.

For general use, these disadvantages are quite outweighed by the fact that the whole record of the cooling of an alloy can be obtained in one curve, but for special purposes, such

* Alloys Research Committee, 3rd and 4th Reports: Proc. Inst. Mech. Eng. 1895, p. 269, and 1897, p. 67.

Fig. 2.


COOLING CURVES
OF ALUMINIUM, ZINC AND
COPPER-TIN ALLOYS
A (ALUMINIUM)
D
C
B
Z (ZINC)
TIME
800°
700°
600°
500°
400°
300°
200°

as obtaining accurate measurements of melting- or boiling-points and investigating the exact shape of the "cooling-curve" of a metal or alloy at the temperature of freezing or of other molecular change, a modification of the method becomes necessary. In order to obtain large-scale records of these changes, Roberts-Austen employed a more sensitive galvanometer, its zero position being adjusted by trial until the spot of light from its mirror fell upon the photographic plate when the thermo-couple was at the particular temperature at which a record of the cooling was desired. The zero position of the enlarged record was often many feet from the plate, and, as the galvanometer had to be deflected through a large angle, its indications became untrustworthy, and it was necessary to employ simultaneously a second ordinary galvanometer to give a measurement of the temperature at which the enlarged record was taken. The two galvanometers were connected in parallel and arranged one behind the other *, so that they both recorded their deflexion on the same moving plate. A record taken in this way is given in fig. 3.

It represents the melting and solidification of an ounce of pure gold; the thermo-junction was immersed in the molten gold without any covering, and had therefore scarcely any lag other than that produced by the galvanometer itself. The curves A and B were produced by the less sensitive galvanometer, and the scale on the left has been obtained by measuring the ordinates of these curves from the datum on the original plate. The scale for the more sensitive records C and D is 22 times as great as the other, and it is obtained from the latter by comparing the slopes of the two records.

The curves A and C represent the melting, and the curves B and D the solidification of the gold. The irregularities in the melting record probably indicate the fall of pieces of still solid gold into the molten mass. On account of this irregularity *cooling*-curves are usually employed, instead of heating-curves, in investigating the constitution of metals and alloys. The curve representing the melting of the gold is nearly 1° higher on the scale than the curve representing

* Roberts-Austen, Proc. Inst. Mech. Eng. 1895, p. 243, and fig. 4, pl. xxxvii.

its solidification, and it would appear that the true melting-point of a pure metal is indicated most nearly by that part of the cooling record where the curve first becomes approximately horizontal.

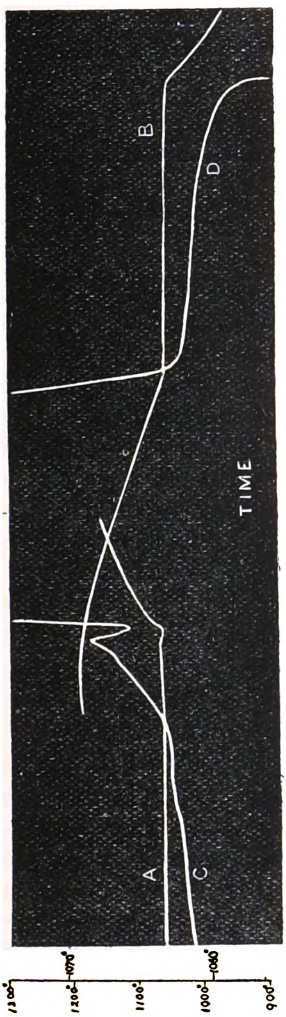


Fig. 3.—Melting and Solidification of Gold.

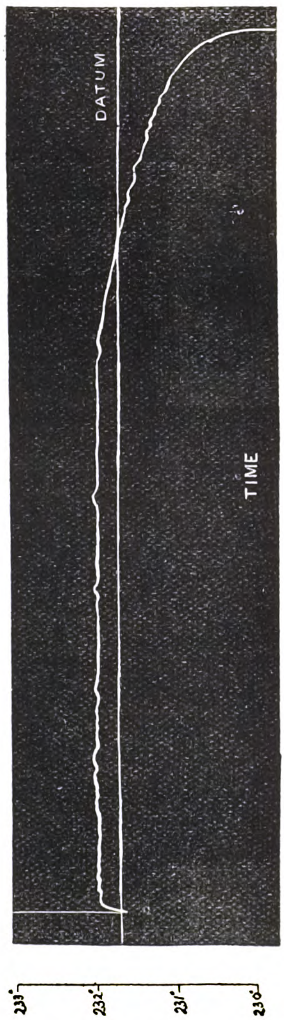


Fig. 4.—A record of the Solidification of Tin.

The author considered two methods for avoiding the large deflexion of the sensitive galvanometer and thus rendering its indications more trustworthy :—(1) that of maintaining the

"cold junction" of the thermo-couple at some high temperature, such as the boiling-point of sulphur, thus measuring *differences* between two high temperatures; and (2) that of balancing the greater part of the E.M.F. of the thermo-couple by a potentiometer method, recording the small remainder by means of a sensitive galvanometer on a moving photographic plate. The latter method was adopted as being much more convenient and generally applicable.

The New Form of Recording Pyrometer.

The Potentiometer.—As the ordinary stretched-wire form of potentiometer, with sliding contacts, would have been unsuitable for the purpose, a special instrument was constructed with plug contacts by means of which any number, from 1 to 99, of equal units of E.M.F. could be inserted in the galvanometer circuit.

The potentiometer, which is shown at B in fig. 1, consists of four sets of resistance-coils; a, a' are sets of nine resistances of 2 ohms each, and b, b' of 0·2 ohm each, while c is a set of 1000 ohm coils which are placed in the battery circuit. The coils a, b are always in circuit with the galvanometer G and thermo-couple Tc. The battery current flows through the set of resistances c and through parts of the resistances in a' and b', and the circuit is completed by the resistances in a and b which lie between the plugs P, P′.

The terminals of the resistances in a, a' are numbered 0, 10, 20, &c. to the left, and in b, b' they are numbered 0, 1, 2, &c. to the right, so that the readings opposite the two plugs give the number of units of E.M.F. in the galvanometer circuit. This arrangement, which involves the use of a larger number of resistance-coils than is usual in potentiometers, was adopted in order that the resistances of the galvanometer circuit and of the battery circuit might remain the same for all positions of the plugs P, P′. The relative resistances of the coils a, b were carefully measured, and a table was calculated giving their values in terms of the mean unit. Changes in the temperature of the potentiometer should not affect the E.M.F. included in the galvanometer circuit, provided that all the coils are at the same temperature. As it was feared that the high-resistance coils c would become

heated by the passage of the current to a greater degree than the low-resistance coils, the box B was provided with a zinc lining and the coils were so arranged they might all be immersed in paraffin, which could be kept in circulation by means of a stirrer. Experiments on the relative resistances of the high and low resistance-coils failed to show that any change was produced by the passage of the extremely small potentiometer current, and the use of the paraffin was, therefore, not resorted to.

The current flowing through the potentiometer is maintained by means of a Clark cell g with large electrodes, each electrode being 20 sq. in. in area. The Clark cell is placed in series with a resistance of 2500 ohms. Under these circumstances, as has been shown by S. Skinner *, the current produced is remarkably constant, and there is also this advantage in the use of the Clark cell, that its E.M.F. when producing a current can be easily compared with that of standard Clark cells of the usual form. The use of Clark cells for standards has necessitated a large temperature correction; but as the cells were placed in a thick wooden box D, and their temperature only rose 2 or 3 tenths of a degree during the day, the rate of change of temperature was sufficiently slow to render this correction fairly legitimate. The E.M.F. of the large Clark cell g is compared with that of the standards h, h, h by means of the auxiliary potentiometer p, which is connected in series with a dry cell j and resistance r (the necessary connexions can easily be traced on the figure). This potentiometer measures the small difference between the E.M.F. across the terminals of the main potentiometer and the E.M.F. of each of the standards h, h, h. By an arrangement which is not shown in the figure, the E.M.F. across the terminals of p is obtained in potentiometer units, and so the readings on p can be converted into a correction to be applied to the readings in B.

The Galvanometer.—The sensitiveness of the pyrometer depends on the sensitiveness of the galvanometer and its constancy of zero. The difficulty of obtaining a sufficiently sensitive galvanometer was increased by the necessity for introducing the resistance (20 ohms) of the potentiometer.

* Phil. Mag. Sept. 1894, p. 271; Proc. Phys. Soc., Feb. 1895.

In experimenting with sensitive suspended-coil galvanometers, great difficulty was experienced in finding one which would return to its zero position after being deflected. This difficulty was at first attributed to the viscosity of the phosphor-bronze strip used to suspend the coil, but it was found that the coil would vibrate when suspended by a silk fibre between the poles of a permanent magnet, a fact which showed that the coil itself was magnetic and that the change in zero was largely due to a change in the direction of the induced magnetism.

A galvanometer, designed to be as free as possible from this defect, was made by Dr. Muirhead; the coil was shuttle-shaped, instead of being hollow as in the ordinary form, and was suspended by a strip at the top; the connexion at the bottom was made by a fine spiral. This instrument returned to its zero-reading fairly well after small deflexions, but for large deflexions it was less reliable than the older form; it was, moreover, much more susceptible to vibrations, a fact which greatly reduced its usefulness, as it was impossible to obtain a reasonably smooth photographic record of its readings. All the records shown were obtained by the open-coil galvanometer.

The Recording Apparatus, shown at H in fig. 1, consists of a photographic plate I, mounted on a float F, which slowly rises as water is admitted into the cylinder H. Vertical glass rods at the sides of the cylinder act as guides to prevent rotation or lateral movement of the float. The whole is covered by a hood K, which can readily be raised by means of the pulley and counterpoise. It is supported by a "hole, slot, and plane," on the fixed board R, with which it makes a light-tight joint. A slit S, which forms part of the hood, permits only horizontal beams of light to reach the plate; consequently it is possible to have the room fully illuminated without danger of fogging the plate. The source of light L is an ordinary glow-lamp enclosed in a wooden box. A brass tube, with a rectangular diaphragm at the end nearest the lamp, cuts off all the light except that from a selected piece of its vertical filament. Light from this filament is reflected by means of a plane mirror on the galvanometer, and focussed on the photographic plate by a lens in front of the galvanometer.

This method of focussing the light, which was suggested by Prof. Boys, enabled a fine line of light to be obtained for recording, notwithstanding the fact that in order to obtain records on a sufficiently large scale the galvanometer was placed 16 feet from the photographic plate. This line of light passing through a horizontal slit immediately in front of the plate produces a square spot on the plate. The glow-lamp is very convenient for ordinary work, but the light is insufficient for taking rapid records, and it becomes necessary, in taking such records, to employ a lime-light.

The Thermo-Couple consists of two wires, one of platinum and the other of platinum alloyed with 10 per cent. of rhodium or iridium. The wires are fused together at the end and inserted for protection into a thin fire-clay tube. Each wire was insulated from the other, except at the junction, by means of a slip of mica. The free ends of the wires were soldered to the leads from the recorder, and these "cold junctions" were kept in ice, or in cold water the temperature of which was noted. More recently an hypsometer has been used, as shown at M in the figure, thus keeping the "cold junctions" at the temperature of boiling water, which has the advantage of being more constant than the temperature at which cold water can readily be maintained; the hypsometer also requires less attention than a "cold junction" involving the use of ice. The interval of temperature to be measured is also reduced, and errors in the E.M.F. are decreased in the same proportion.

Method of taking a Record.—The photographic plate is first placed on the float F, using a red light in the room, and a cover is placed over the slit to keep out the spot of light. The galvanometer is short-circuited by the shunt-box *e*, and the thermo-couple, in its clay tube, is placed in the metal or alloy, N, which has been melted in a gas-furnace. The flame is then either turned out or reduced in size, so that the furnace may cool slowly. The plugs P, P′ are placed in position in the potentiometer, and the shunts are gradually removed from the galvanometer as the balance is obtained. When the galvanometer has its full sensibility, the spot of light travels almost across the photographic plate every time that the position of P′ is changed in order to maintain the balance as

the furnace cools down. This change is effected by placing another plug immediately to the left of P′ before the latter is removed; when it becomes necessary to move the plug P the galvanometer is short-circuited. When the metal has cooled to the point at which a record is desired, the water-clock is started, and the cover removed from the slit to admit the spot of light from the galvanometer to the photographic plate. As the movement of the spot of light can be seen on a scale immediately above the slit, it is easy to follow the progress of the experiment without interfering with the record. As the metal cools down and the photographic plate rises, successive tracks of the spot of light may be obtained on the plate, each track corresponding to a particular position of the plugs. When the cooling has been carried sufficiently far, the galvanometer is disconnected from the thermo-couple and short-circuited by a contact-piece at *f*, and a datum-line is run on the plate to indicate the zero position of the galvanometer.

It is also necessary

(1) To take the barometer-reading to determine the temperature of the "cold junction."

(2) To obtain the value of the potentiometer E.M.F. in terms of the standard Clark cell by means of the auxiliary potentiometer.

(3) To calibrate the auxiliary potentiometer.

(4) To read the temperature of the Clark cells by means of the thermometer *t*.

(5) To determine the relation between the deflexion of the galvanometer and the potentiometer-unit.

Fig. 4 is a reproduction of a record, obtained in this way, of the solidification or freezing of tin. The metal was contained in a small clay crucible inside a cast-iron chamber heated externally, so that very slow and regular cooling could be obtained.

The metal was stirred during the solidification in order to promote uniformity of temperature and to prevent surfusion. It will be noticed that the temperature remained practically constant until the greater part of the metal had solidified.

Records have also been taken on the same scale of the

freezing of metals having much higher melting-points, such as gold or copper; but the difficulty of obtaining a satisfactory curve is greatly increased.

Discussion of the Observations.

In order to discuss the theoretical bearing of the readings of the thermo-couples it is necessary to know the temperatures to which they correspond. Although measurements of high temperatures have been made with the air-thermometer, they are not given in the present research. The results of different experimenters have nevertheless determined the melting-temperatures of silver, gold, and copper, with an error of probably less than 10°; the relative temperatures being known much more accurately. For the purposes of the mathematical treatment of the results, the melting-temperature determinations of Heycock and Neville may be taken. Although their measurements of the melting-temperatures of silver, gold, and copper depend on a somewhat severe extrapolation of a formula, their determinations of the *relative* melting-points of these metals are probably very accurate, and their results agree fairly well with those obtained by the direct air-thermometer observations of Holborn and Wien.

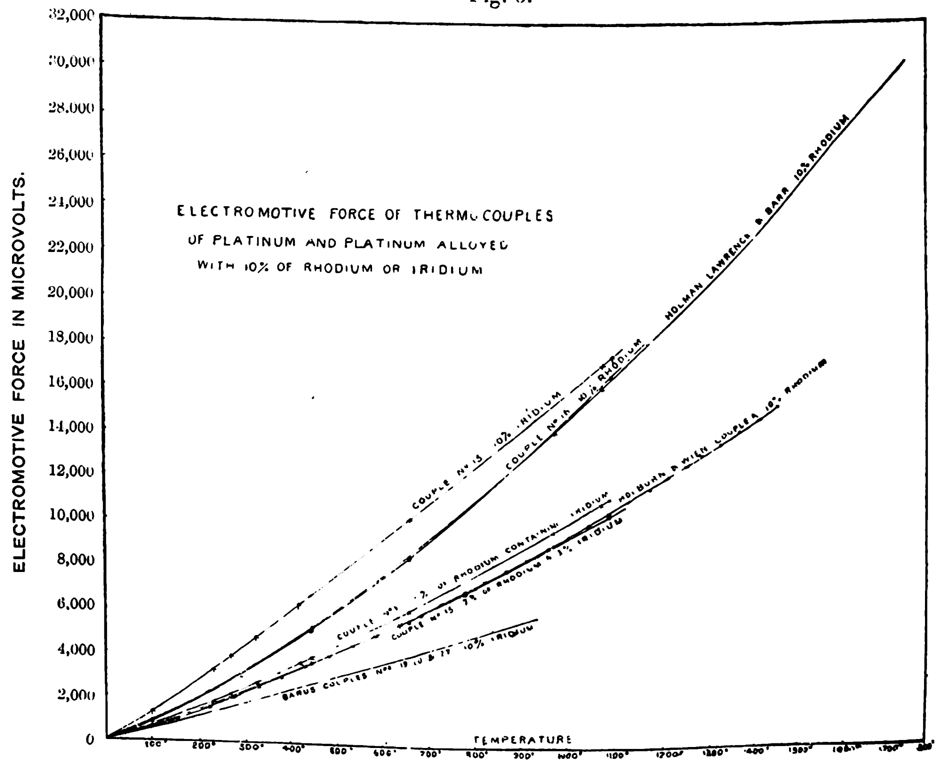
In fig. 5 the results obtained by the author for different thermo-couples are plotted in the usual way; the co-ordinates representing the temperature of the hot junction (t), and the observed E.M.F. of the couple (E), the cold junction being supposed to be at 0° C.

Curves representing some of the observations of Barus, Holborn and Wien, and Holman, Lawrence, and Barr have also been plotted. An examination of figure 5 shows that very serious discrepancies exist between the indications of couples having the same nominal composition. Thus the thermo-electric power of the 10 per cent. rhodium thermo-couple of Holborn and Wien was only about 3/5 of that obtained by Holman, Lawrence, and Barr; and the 10 per cent. iridium couple employed by the author had about $2\frac{1}{2}$ times the thermo-electric power of a 10 per cent. iridium alloy examined by Barus. Through the kindness of Colonel Edward Matthey, the author has been able to investigate the causes of these discrepancies, which are too large to be

attributed merely to the accidental differences in the percentage of rhodium or iridium in the platinum alloy used.

Two 10 per cent. rhodium alloys were prepared and drawn into wires from which the thermo-couples, Nos. 15 and 16, were made up with a return wire of pure platinum common to

Fig. 5.



both. The rhodium used in the composition of No. 15 was part of a commercial sample and was found to contain no less than 30 per cent. of iridium, so that the alloy consisted of 90 per cent. platinum, 7 per cent. rhodium, and 3 per cent. iridium; No. 16, on the other hand, contained 10 per cent. of pure rhodium. Observations made with these couples reveal the unexpected fact that, although the 10 per cent. iridium alloy is rather more powerful than the rhodium alloy,

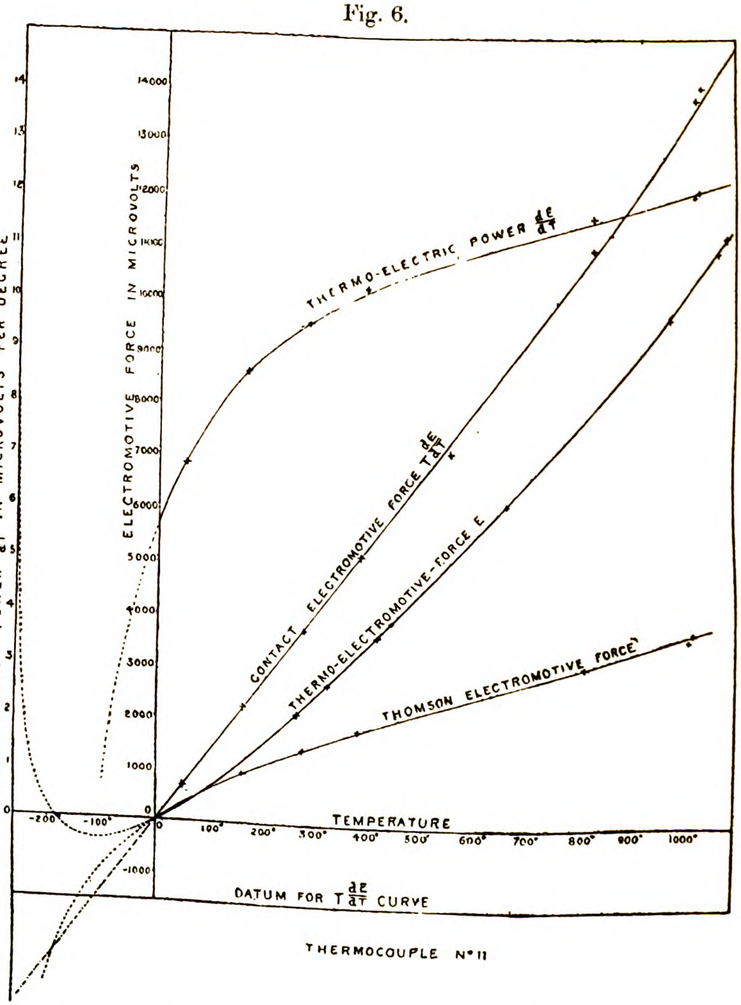
the partial substitution of iridium for rhodium in the latter alloy very materially lowers its thermo-electric power when compared with platinum. This result is of interest, as it suggests that the change in the thermo-electric power of a metal depends on the extent to which it is saturated with the alloying metal; 10 per cent. of either rhodium or iridium would more completely saturate the platinum than would 10 per cent. of a mixture of the two metals, in the same way that a single salt more completely saturates a liquid than the same weight of a mixture of two salts of equal solubility. The curve marked "Barus, 10 per cent. iridium" was obtained by adding the mean results obtained by him with couples 19 and 20, consisting of "hard" platinum and platinum with 5 per cent. of iridium, to those obtained with couple 27, consisting of platinum with 5 per cent. of iridium and platinum with 10 per cent. of iridium. The use of "hard," *i. e.* impure platinum in these alloys probably accounts in part for the smallness of the observed thermo-electric power. A comparison of his couples 22 and 35 shows that the E.M.F. at 930° would have been about 1200 microvolts higher if "soft" platinum had been used: this, however, does not materially reduce the difference between his results and those obtained by the author with couple 13, and it appears probable that the iridium he employed contained a considerable percentage of rhodium or other metal of the platinum group.

Although the curves in fig. 5 differ so widely from one another in direction, there is a general agreement as regards their form. All the curves are convex towards the scale of temperature and their curvature decreases as the temperature rises. The 10 per cent. iridium curves are decidedly straighter in their lower parts than the 10 per cent. rhodium curves, so that their indications are more nearly proportional to the temperature.

The usual method of presenting thermo-couple observations has been adopted in plotting the curves in fig. 5, but it is evident that a more critical method is necessary; for although the uncertainties in the actual temperatures of points would be quite visible on these curves, the observations themselves can be made with an accuracy of about 1 in 10,000. Very good results are obtained by plotting the first differential of

the original curve; that is a curve the ordinates of which represent the values of $\frac{dE}{dt}$ corresponding to the values of t indicated by the abscissæ.

Fig. 6.



This method can only be applied when a number of melting- or boiling-points have been obtained, so as to enable the value $\frac{dE}{dt}$ to be found for a sufficiently large number of points.

For rough purposes, these values are obtained by dividing the difference between the observed E.M.F.'s at two adjacent melting-points by the difference in the assumed temperatures, and plotting the result at the mid-point. A more accurate curve may be obtained by a graphic method which it is unnecessary to describe here.

In fig. 6, the thermo-electric power or $\frac{dE}{dT}$ curve has been obtained in this manner from the curve representing the thermo-electromotive force E of couple No. 11.

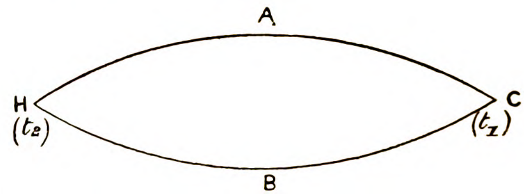
Small errors, either in the assumed temperatures of the melting-points, or in the observed values of the E.M.F. for these temperatures, are thrown into prominence by this method of plotting; errors of a few tenths of a degree in the relative observations of the melting-points of tin, bismuth, or lead, or of one or two degrees in the melting-point of aluminium, make very distinct breaks in the curve, and although the observed values of E are all seen to lie closely on the E.M.F. curve, the calculated values of $\frac{dE}{dT}$ show in some cases a wide divergence from the smooth mean curve. Thus, for example, the observed melting-point for aluminium must have been several degrees lower than the assumed temperature, unless the curve can be admitted to have a great dip at about 550°. Records of the solidification of the aluminium employed did not show the steady freezing-temperature which is characteristic of a pure metal, and the author is inclined to attribute the low reading for aluminium (see Table II., p. 127) to the impurity of the metal rather than to irregularity of the indications of the thermo-couple.

Theory of the Thermo-Couple.

The practical value of any pyrometer depends largely on the possibility of expressing, accurately and simply, the relation between the reading of the pyrometer and the temperature to which it is exposed. In the case of the thermo-couple this has been attempted by many experimenters, partly empirically, and partly from theoretical considerations.

The simple thermo-couple HACB, consisting of two wires A and B of different metals or alloys, has its junctions H

and C at temperatures t_2 and t_1 respectively; four separate E.M.F.'s are produced: (1) a true contact E.M.F. at H, and (2) another at C, due to the difference in the nature of the two wires (the Peltier E.M.F.), and (3 and 4) an E.M.F. distributed along each of the wires A and B, due to the difference between the temperatures of different parts of the same wire (the Thomson E.M.F.).



The observed E.M.F. is, in reality, the algebraic sum of the difference between the E.M.F.'s at H and C, and the difference between the E.M.F.'s in A and B.

Lord Kelvin has shown that it is possible, by means of the second law of thermo-dynamics, to distinguish between some of these constituents of the observed E.M.F., for the thermocouple may be regarded as a reversible engine.

Let σ be the "specific heat of electricity" in the metal A, that is, the amount of heat (measured in ergs) absorbed by the passage of one unit of electricity (or by unit current in unit time) through a rise in temperature of one degree; and let σ' be the specific heat of electricity in B.

Let H be the coefficient of the Peltier effect, that is, the amount of heat absorbed at the junction by unit current in unit time passing from A to B. Then if T_1 be the temperature of the cold junction, and T_2 that of the hot junction, measured from the absolute zero, the heat absorbed in the wires A and B in unit time by the passage of a current I will be $I(\sigma-\sigma')(T_2-T_1)$ if the temperatures T_2 and T_1 are nearly equal, so that the values of σ and σ' may be taken for a mean-temperature T.

It then follows, from the second law of thermo-dynamics, that

$$\frac{IH_2}{T_2}-\frac{IH_1}{T_1}+\frac{I(\sigma-\sigma')(T_2-T_1)}{T}=0,$$

and as
$$E = H_2 - H_1 + \int_{T_1}^{T_2} (\sigma - \sigma')dT,$$

it follows that
$$H = T\frac{dE}{dT},$$

and
$$E = T\frac{dE}{dT} + \phi(T) - C. \quad . \quad . \quad . \quad . \quad (A).$$

In this equation $T\frac{dE}{dT}$ is the E.M.F. at the hot junction, C is the E.M.F. at the cold junction, and $\phi(T)$ is $\int_{T_1}^{T_2}(\sigma-\sigma')dT$, the difference between the Thomson effects in the two wires.

No theoretical investigations have as yet determined the form of $\phi(T)$, so that it is impossible to obtain, from purely theoretical considerations, an equation connecting the temperature and the electromotive force of the couple.

Equation (A) does, however, enable us to calculate the values of the Peltier and the Thomson E.M.F.'s from a series of data connecting the observed E.M.F. and the temperature of the thermo-junction, and the results of such a calculation for the thermo-couple No. 11 are given in fig. 6.

The contact or Peltier E.M.F. of the thermo-junction is represented in the equation by $T\frac{dE}{dT}$, and may be deduced from the thermo-electric power or $\frac{dE}{dT}$ curve by multiplying each value of $\frac{dE}{dT}$ by the corresponding absolute temperature T.

The $T\frac{dE}{dT}$ curve, embodying the results of these calculations, has been plotted in the figure, the ordinates are measured from a datum-line placed at a distance below the temperature-scale equal to the E.M.F. across the cold junction; so that its ordinates from the temperature-scale give the difference between the E.M.F.'s at the hot and cold junctions. The difference between this curve and the E.M.F. curve E represents the resultant of the Thomson effects in the two wires. The curve marked "Thomson E.M.F." obtained in this way has been plotted for convenience *above* the temperature-scale

line, but it must be remembered that this E.M.F. is opposed in direction to the resultant of the contact E.M.F.'s.

It will be noticed that the curve representing the contact E.M.F., or $T\frac{dE}{dT}$, is practically a straight line. In fact, the divergences at 1000° only correspond to an uncertainty of about 2° in the assumed temperatures.

If, then, we assume that

$$T\frac{dE}{dT} = aT + b,$$

we obtain by integration $E = aT + b \log T + c$.

This equation requires three relations between E and T to determine the values of the three constants a, b, and c, and these can be accurately obtained for the temperatures 0°, 100°, and the boiling-point of sulphur (444°·53).

The temperatures given in the last column of Table II. (p. 127), were obtained in this way from the Oct 1896 results of couple 11. These temperatures really depend on an extrapolation beyond 445° of the empirical formula given above, a formula which appears to be correct in the case of this thermo-couple as far as it has been possible to verify it. The dotted lines in fig. 6 are the continuations of the curves at temperatures below 0°. They depend on the assumption that the formula holds good at these temperatures, and show that at the absolute zero the E.M.F. would become infinitely great, and that the contact E.M.F. would have a negative value of about 200 micro-volts: it has been stated* that the latter should be zero, as otherwise further cooling could be effected by the passage of a current. The $T\frac{dE}{dT}$ curve for a platinum-lead thermo-couple, calculated from figures given in Dewar and Fleming's † paper on thermo-electricity at low temperatures, is straight between 100° and 0°, but below this temperature it shows a tendency to curve in the direction required to reduce its value to nothing at the absolute zero.

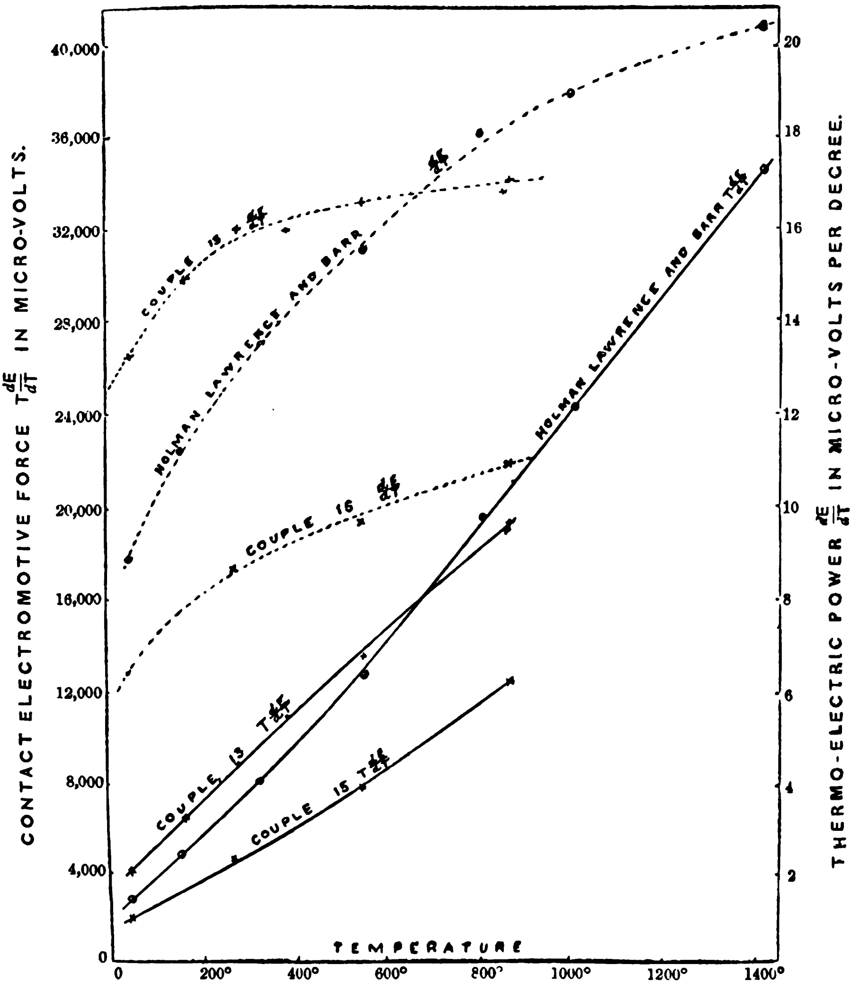
The theoretical simplicity of the formula given above, which

* Burton, Phil. Mag. vol. xxxviii. 1894, p. 66.

† Phil. Mag. vol. xl., July 1895, p. 95.

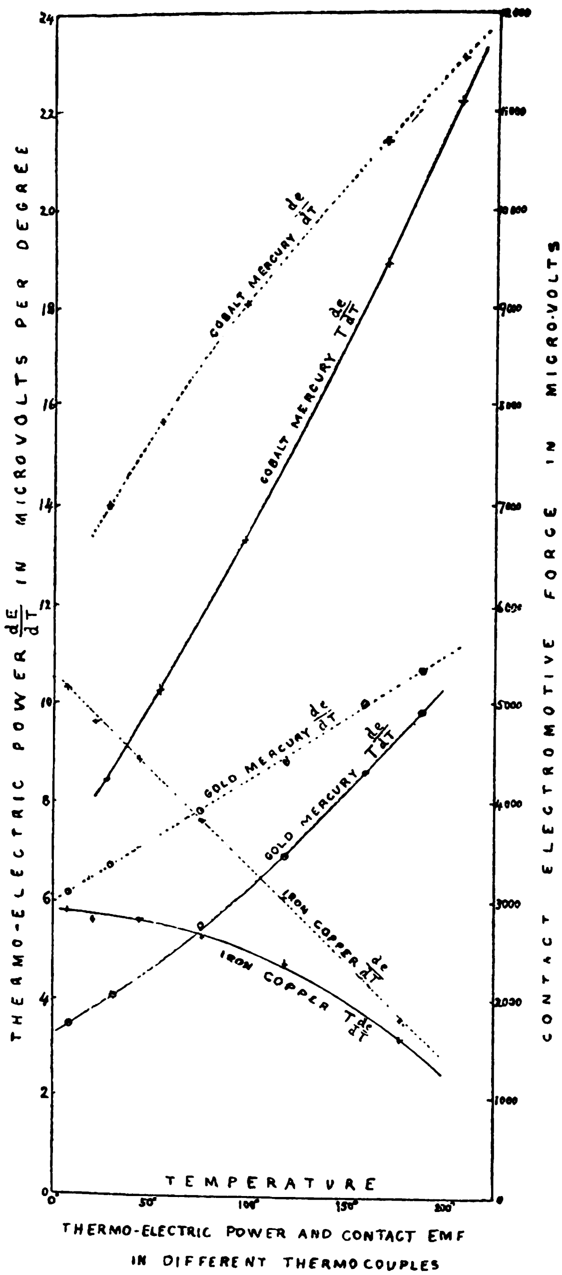
involves a linear relation between the temperature of a junction and the Peltier E.M.F., makes it worth while to consider how far this relation is true in the case of other thermo-couples. In fig. 7, which has been plotted with this object, the

Fig. 7.



THERMO-ELECTRIC POWER AND CONTACT E.M.F.
IN PLATINUM THERMO-COUPLES.

Fig. 8.


THERMO-ELECTRIC POWER $\frac{dE}{dT}$ IN MICROVOLTS PER DEGREE
CONTACT ELECTROMOTIVE FORCE IN MICRO-VOLTS
TEMPERATURE
COBALT MERCURY $\frac{de}{dT}$
COBALT MERCURY $T\frac{de}{dT}$
GOLD MERCURY $\frac{de}{dT}$
GOLD MERCURY $T\frac{de}{dT}$
IRON COPPER $\frac{de}{dT}$
IRON COPPER $T\frac{de}{dT}$
THERMO-ELECTRIC POWER AND CONTACT EMF
IN DIFFERENT THERMOCOUPLES

dotted lines represent the values of $\frac{dE}{dT}$, and the continuous lines those of $T\frac{dE}{dT}$, for the couples 13, 15, and 16, and for the couple used by Holman, Lawrence, and Barr.

None of the curves representing the contact E.M.F. are so nearly straight as in the case of couple 11; but all these curves are approximately straight, whilst those representing the values of $\frac{dE}{dT}$ are all strongly curved.

This is of great interest in connexion with the parabolic formula investigated by Tait. Tait's expression for the E.M.F. of a thermo-couple follows from equation (A) if the assumption be made that the specific heat of electricity is proportional to the absolute temperature; but the parabolic formula is really empirical, and depends on the observations that the curves representing the values of $\frac{dE}{dT}$ are approximately straight in the case of the thermo-couples for which the formula was constructed. The formula of Tait and the formula developed in this paper are both empirical, and are both based on theoretical assumptions. In figures 6, 7, and 8, the extent to which these formulæ hold can be seen at a glance. The parabolic formula depends on the straightness of the $\frac{dE}{dT}$ curve, which is dotted in figs. 7 and 8, whilst the logarithmic formula depends on the straightness of the $T\frac{dE}{dT}$ curve.

The curves which are plotted in fig. 8 are calculated from results given in a paper by K. Noll * on "The Thermo-electricity of Chemically Pure Metals." They show that thermo-couples composed of copper, iron, gold, and mercury follow Tait's law very closely over the somewhat limited range of temperatures for which data are given, and other couples not plotted also show approximately straight lines for the values of $\frac{dE}{dT}$. A couple composed of cobalt and mercury, and another composed of platinum and copper the results of which are not plotted, show both lines curved in opposite directions, a fact

* Wied. *Ann.* 1894, p. 874.

TABLE I.

Electromotive Force in Micro-volts of Thermo-couples. Cold junctions at 0° C.

Substances.	Assumed temperatures.	Couple 11.				Couple 13.		Couple 15.	Couple 16.
		March 1896.	Oct. 1896.	March 1897.	Nov. 1897.	1st set, 1898.	2nd set, 1898.	Feb. 1898.	Feb. 1898.
Water boils	100°	690	686	......	......	1324	1323	642	858
Tin melts	231·9	1811	1820	......	1784	3320	......	......	......
Bismuth melts ...	268	2171	2160	......	2117	3929	......	......	......
Lead melts	326	2723	2718	......	......	4840	......	......	......
Zinc melts	419	3654	3652	......	......	6331	......	......	......
Sulphur boils ...	444·53	......	3927	......	3894	6733	6713	3621	5245
Aluminium melts	654·5	......	6138	......	......	10211	......	5042	8528
Silver melts	960·7	......	9686	9669	......	......	......	......	......
Gold melts.........	1061·7	10941	10894	10938	......	......	17056	......	......
Copper melts......	1080·5	11192	11133	11132	11106	......	17427	10318	16536

Note.—The changes in the E.M.F. of couple 11 during two years correspond to a change of about 5° at 1100°. They were principally caused by prolonged heating in fire-clay insulators in a reducing atmosphere before the Oct. 1896 and Nov. 1897 observations. Under normal conditions the changes are very much smaller.

Table II.

Comparison of the Temperatures of the Melting-points of Metals afforded by recent research.

Authority.	Heycock & Neville.	Holborn & Wien.	Holman, Lawrence, & Barr.	D. Berthelot.	The Author.
Date of publication.	1895.	1895.	1896.	1898.	1898.
Instrument	Resistance Pyrometer.	Thermo-couple.	Thermo-couple.	Interference Method	Thermo-couple no. 11.
Calibration data.	0° 100° 444°·53	Porcelain air-thermometer.	0° 444°·53 100° 1072° 218°·2	Expansion of air.	0° 100° 444°·53
Tin	231°·9	......	......	......	232°·1
Bismuth......	......	......	......	......	268·4
Lead	......	......	......	......	325·9
Zinc	418·96	......	......	......	418·2
Aluminium .	654·5	......	660°	......	649·2
Silver	960·7	968°	970	962°	961·5
Gold	1061·7	1072	(1072)	1064	1062·7
Copper	1080·5	1082	1095	......	1083·0

which suggests that thermo-electrically there may be two classes of metals:—(1) the ordinary metals, for which the $\frac{dE}{dT}$ curves are straight; and (2) the platinum metals, together with a few other metals such as nickel and cobalt, for which the $T\frac{dE}{dT}$ curves are straight. A couple composed of one metal from each class would of course give both lines curved. On the other hand, it is unlikely that there would be any sharp division of this kind, and both formulæ may be particular cases of a more general one.

In view of the curvature exhibited by the $\frac{dE}{dT}$ curves of thermo-couples composed of platinum and platinum alloyed with iridium or rhodium, there can be no object in applying

the parabolic formula to their indications, though this has been done by several experimenters.

In conclusion, the author wishes to express his thanks to Prof. Roberts-Austen for the helpful interest he has taken in the research.

Discussion.

Dr. Chree discussed the curves and asked how far stirring affected the results; he was inclined to think that stirring was a mistake.

Mr. A. Campbell enquired whether the galvanometer kept its zero sufficiently well throughout the tests.

Mr. Stansfield, in reply, said he had also come to the conclusion that stirring was a mistake; it was a mistake to use a large quantity of metal. The pyrometers were sensitive to about a tenth of a centigrade degree. He had experienced great difficulty with regard to the zero of the galvanometer.

XVIII. *On Dynamical Illustrations of Certain Optical Phenomena. By* Professor J. D. EVERETT, *F.R.S.**

CONTENTS.

§ 1. Introductory.
§ 2. Chain of vibrating particles.
§ 3. Simple travelling undulation.
§ 4. Geometrical theorem.
§ 5. One particle of chain constrained to given motion.
§ 6. Constraining force and its work.
§ 7. Stokes's application to fluorescence.
§ 8. Stationary vibration of chain.
§ 9. Mutually influencing pendulums.
§ 10. Double pendulum.
§ 11. Case of small lower particle.
§ 12. Applications by Kelvin and Rayleigh.
§ 13. Case of lengths nearly equal.
§ 14. Sellmeier on fluorescence.

§ 1. IN studying the attempts which have been made to illustrate certain optical phenomena by analogies drawn from chains of particles and from mutually influencing pendulums, I have been impressed with the difficulty of obtaining accurate information respecting the dynamical results in question, and the proofs on which they rest. I have taken some pains to find brief and simple methods of establishing the most important properties, and have here arranged both proofs and results in a form convenient for reference.

In discussing the chain of particles, I have not had recourse to theorems in the Calculus of Finite Differences, but have imitated the methods employed in Lord Kelvin's Baltimore Lectures; and in this connexion I have hit upon a curious geometrical and kinematical theorem which I believe is new.

In dealing with mutually influencing pendulums, whether suspended side by side or one from the other, I have simplified the usual investigation, by first assuming the motion to take place in a fundamental mode, and afterwards discussing combinations of such modes. The deduction of the properties of

* Read March 11, 1898.

the double pendulum here given is, I believe, the fullest to be found anywhere.

A "fundamental mode of vibration" may be defined as one in which *all the particles have simple harmonic vibrations with the same period and either identical or opposite phases.* The resolvability of the most general small permanent vibrations of a system into modes of this simple character is a well-known proposition of abstract dynamics. Anyone who objects to assuming it may regard our plan of operations as a tentative method which is justified when it leads to solutions containing the proper number of arbitrary constants.

§ 2. Suppose a series of equal particles attached at equidistant points to a uniform elastic string of negligible mass. For simplicity we shall ignore gravity, and suppose the only forces acting on each particle to be the tensions of the two portions of string between which it lies. Then the series, when in equilibrium, will lie in a straight line, with uniform tension throughout the whole string. Let F denote this uniform tension, and a the common distance between the particles. Then, in the case of small transverse vibrations in one plane, if y_1, y_2, y_3 denote the displacements of three consecutive particles, and the mass of each particle be M, we shall have

$$\ddot{y}_2 = -\frac{F}{M}\frac{y_2-y_1}{a} - \frac{F}{M}\frac{y_2-y_3}{a},$$

or, putting μ for F/Ma,

$$\ddot{y}_2 = -\mu(y_2-y_1) - \mu(y_2-y_3) = -\mu(2y_2-y_1-y_3) \quad . \quad . \quad (1)$$

These formulæ are also applicable to longitudinal vibrations, if we make F denote Young's modulus multiplied by the cross section.

§ 3. When a simple harmonic undulation is running along the chain of particles, we shall have

$$y = A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

y denoting the displacement of the particle whose undisturbed abscissa is x, λ the wave-length, t the time, T the period, and A the amplitude.

Equation (1) applied to the mth particle gives

$$\ddot{y} = -A\mu\left\{2\sin 2\pi\left(\frac{ma}{\lambda}-\frac{t}{T}\right)-\sin 2\pi\left(\frac{m-1}{\lambda}a-\frac{t}{T}\right)-\sin 2\pi\left(\frac{m+1}{\lambda}a-\frac{t}{T}\right)\right\}$$

$$=-2A\mu\left\{\sin 2\pi\left(\frac{ma}{\lambda}-\frac{t}{T}\right)-\sin 2\pi\left(\frac{ma}{\lambda}-\frac{t}{T}\right)\cos\frac{2\pi a}{\lambda}\right\}$$

$$=-4A\mu\sin 2\pi\left(\frac{ma}{\lambda}-\frac{t}{T}\right)\sin^2\frac{\pi a}{\lambda}=-4\mu y\sin^2\frac{\pi a}{\lambda}.$$

But by direct differentiation of (2), $\ddot{y}=-\left(\frac{2\pi}{T}\right)^2 y$.

Hence

$$\left(\frac{2\pi}{T}\right)^2=4\mu\sin^2\frac{\pi a}{\lambda};\quad \frac{1}{T^2}=\frac{\mu}{\pi^2}\sin^2\frac{\pi a}{\lambda}. \quad . \quad . \quad (3)$$

Let V denote the velocity of propagation; then

$$V^2=\frac{\lambda^2}{T^2}=\frac{\mu\lambda^2}{\pi^2}\sin^2\frac{\pi a}{\lambda}=\mu a^2\left(\frac{\sin \pi a/\lambda}{\pi a/\lambda}\right)^2 \quad . \quad . \quad (4)$$

This is equal to μa^2 when $\lambda=\infty$, and diminishes continuously to zero as $\pi a/\lambda$ increases from 0 to π. Hence $a\sqrt{\mu}$ is the velocity of infinitely long waves, zero is the velocity of waves of length a, and there is continuous diminution of velocity between these limits.

The expression $\frac{\mu}{\pi^2}\sin^2\frac{\pi a}{\lambda}$ obtained in (3) for the square of the frequency shows that the frequency vanishes for $\lambda=a$ and $\lambda=\infty$, and attains its maximum value $\sqrt{\mu}/\pi$ when $\lambda=2a$.

The maximum value of the acceleration-factor $-\ddot{y}/y$ or $(2\pi/T)^2$ is 4μ, and is also attained when $\lambda=2a$. Every frequency less than the maximum corresponds to two different values of λ between a and ∞. Calling them λ_1 and λ_2, we have

$$\frac{a}{\lambda_1}+\frac{a}{\lambda_2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It thus appears at first sight that two different modes of undulation correspond to each frequency. The following investigation shows that they are merely two different specifications of one and the same motion of the particles.

§ 4. The two undulations

$$y_1 = \mathrm{A} \sin 2\pi\left(\frac{x}{\lambda_1} - \frac{t}{\mathrm{T}}\right), \quad y_2 = -\mathrm{A} \sin 2\pi\left(\frac{x}{\lambda_2} + \frac{t}{\mathrm{T}}\right), \quad (6)$$

which have equal amplitudes and opposite directions of propagation, will specify the same motion of the particles if $y_1 - y_2$ vanishes for all values of t when x is a multiple of a, say ma. This gives

$$\sin 2\pi\left(\frac{ma}{\lambda_1} - \frac{t}{\mathrm{T}}\right) + \sin 2\pi\left(\frac{ma}{\lambda_2} + \frac{t}{\mathrm{T}}\right) = 0,$$

$$\sin \pi\left(\frac{ma}{\lambda_1} + \frac{ma}{\lambda_2}\right) = 0,$$

$$\frac{a}{\lambda_1} + \frac{a}{\lambda_2} = n\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

n denoting any integer either positive or negative. This includes equation (5) as a particular case.

In like manner the two undulations

$$y_1 = \mathrm{A} \sin 2\pi\left(\frac{x}{\lambda_1} - \frac{t}{\mathrm{T}}\right), \quad y_3 = \mathrm{A} \sin 2\pi\left(\frac{x}{\lambda_3} - \frac{t}{\mathrm{T}}\right),$$

which have equal amplitudes and the same direction of propagation, will be equivalent if

$$\sin 2\pi\left(\frac{ma}{\lambda_1} - \frac{t}{\mathrm{T}}\right) - \sin 2\pi\left(\frac{ma}{\lambda_3} - \frac{t}{\mathrm{T}}\right) = 0,$$

$$\sin \pi\left(\frac{ma}{\lambda} - \frac{ma}{\lambda_3}\right) = 0,$$

$$\frac{a}{\lambda_1} - \frac{a}{\lambda_3} = n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

One and the same undulating motion of the particles can accordingly be represented in a unlimited number of different ways by the uniform motion of a harmonic curve in the direction of $\pm x$. The amplitude, being the amplitude of the motion of a particle, must be the same for all, but the wave-length may have any value consistent either with equation (7) or equation (8).

The simplest specification is obtained by employing the greatest admissible wave-length. This wave-length, which

we will denote by λ_1, will be unique, and will lie between $2a$ and ∞, except when it is equal to $2a$. In general, the order of magnitude will be λ_1, λ_2, λ_3, &c., as defined by

$$\frac{a}{\lambda_2}=1-\frac{a}{\lambda_1},\quad \frac{a}{\lambda_3}=1+\frac{a}{\lambda_1},\quad \frac{a}{\lambda_4}=2-\frac{a}{\lambda_1},\quad \frac{a}{\lambda_5}+2=\frac{a}{\lambda_1},\qquad (9)$$

and so on.

λ_1 is the value obtained by regarding $\pi a/\lambda$ as an angle in the first quadrant.

In the exceptional case $\lambda_1=2a=\lambda_2$, successive particles are in opposite phases, the frequency has the maximum value $\sqrt{\mu/\pi}$, and the acceleration-factor has the maximum value 4μ. The waves in this case can be more simply regarded as stationary.

In the exceptional case $\lambda_1=\infty$, we have $\lambda_2=a=\lambda_3$. The particles are in one straight line, there is no acceleration, and the frequency is zero.

The above investigation establishes the following

Geometrical Proposition.

Through any number of points lying on a harmonic curve and having equidistant ordinates, it is always possible to draw an unlimited number of harmonic curves, having the same amplitude but different wave-lengths. Calling the common distance a, and the wave-length of any curve λ, the curves can be divided into two sets such that for any two of the same set the difference of the values of a/λ is an integer, and for any two of opposite sets the sum is an integer. At the points in question, curves of the same set slope the same way, curves of opposite sets slope opposite ways, and the tangents of the slopes are inversely as the wave-lengths. If all the curves of one set are displaced in one direction, and those of the other set in the opposite direction, along the axis of abscissas, through distances proportional to their wave-lengths, all the curves will still intersect on the original ordinates.

Figs. 1 and 2 illustrate the different possible specifications of the same motion of the particles. Small circles are drawn round the particles to render them more conspicuous. All the curves pass through the centres of these circles. In fig. 1 the values of a/λ for the two curves are $\frac{1}{7}$ and $\frac{6}{7}$, their sum being 1. In fig. 2 the values of a/λ for the four curves are

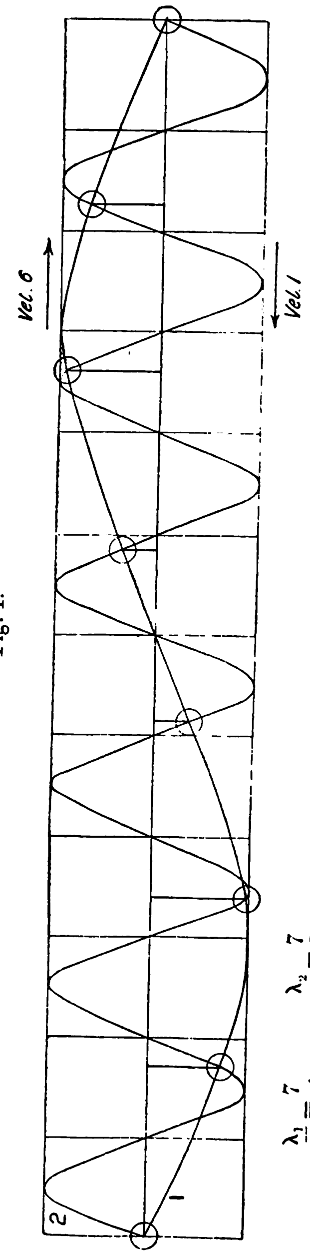


Fig. 1.

$\frac{\lambda_1}{a} = \frac{7}{1}$. $\frac{\lambda_2}{a} = \frac{7}{6}$.

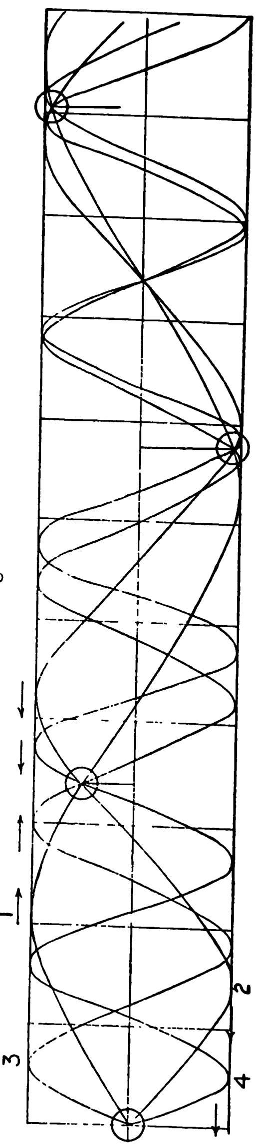


Fig. 2.

$\frac{a}{\lambda_1} = \frac{2}{5}$. $\frac{a}{\lambda_2} = \frac{3}{5}$. $\frac{a}{\lambda_3} = \frac{7}{5}$. $\frac{a}{\lambda_4} = \frac{8}{5}$.

$\frac{2}{5}, \frac{3}{5}, \frac{7}{5}, \frac{8}{5}$; the difference being 1 for the first and third, and for the second and fourth ; and the sum being 1 for the first and second, 2 for the first and fourth, and 2 for the second and third. In fig. 1 the two curves are supposed to be travelling in opposite directions, the flat curve six times as fast as the steep one. In fig. 2, curves 1 and 3 travel one way, and curves 2 and 4 the opposite way.

§ 5. In our chain of particles, if one particle be constrained to simple harmonic vibration with frequency less than the maximum or *critical* frequency $\sqrt{\mu}/\pi$, the permanent state for the chain will be an undulation whose wave-length is given by

$$\sin \frac{\pi a}{\lambda} = \frac{\pi}{\sqrt{\mu}} \frac{1}{\mathrm{T}}, \quad . \quad . \quad . \quad . \quad (3\,\mathrm{A})$$

or a combination of two such undulations travelling in opposite directions ; and this state is one which, if once started, will be kept up by the internal forces of the system.

We shall now investigate the permanent motion of the chain when one of the particles is constrained to simple harmonic vibration of frequency greater than the critical frequency $\sqrt{\mu}/\pi$.

Let y_0 be the displacement at time t of the constrained particle, and $y_1, y_2, y_3 \ldots$ those of the consecutive particles on one side of it. In any " fundamental " or " normal " mode of vibration, the accelerations $\ddot{y}_1, \ddot{y}_2$, &c. are proportional to the displacements y_1, y_2, &c. ; so that we may write

$$\ddot{y}_1 = -\omega^2 y_1, \quad \ddot{y}_2 = -\omega^2 y_2, \quad \text{\&c.}$$

Hence by equation (1) we have

$$\left.\begin{aligned} \omega^2 y_1 &= \mu(2y_1 - y_0 - y_2), \\ \omega^2 y_2 &= \mu(2y_2 - y_1 - y_3), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

and so on.

Hence

$$\frac{\omega^2}{\mu} - 2 = -\frac{y_0}{y_1} - \frac{y_2}{y_1} = -\frac{y_1}{y_2} - \frac{y_3}{y_2} = \text{\&c.} = k \text{ (suppose)}. \quad (11)$$

These equations are satisfied by assuming

$$\left.\begin{aligned} -\frac{y_0}{y_1} = -\frac{y_1}{y_2} &= -\frac{y_2}{y_3} = \text{\&c.} = r, \\ k &= r + \frac{1}{r}, \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

if k is greater than 2, or if $\omega^2 > 4\mu$, or (since ω is $2\pi/T$) if $1/T > \sqrt{\mu}/\pi$.

The same result can be deduced by writing equations (11) in the form

$$-\frac{y_0}{y_1} = k + \frac{y_2}{y_1}, \quad -\frac{y_1}{y_2} = k + \frac{y_3}{y_2}, \quad -\frac{y_2}{y_3} = k + \frac{y_4}{y_3}, \quad \&c.; \qquad (13)$$

whence we obtain for any one of the ratios $-\frac{y_0}{y_1}$, $-\frac{y_1}{y_2}$, &c. the value

$$k - \cfrac{1}{k - \cfrac{1}{k - \&c.}} = r \text{ (suppose)}. \quad . \quad . \quad . \quad . \qquad (14)$$

That is,

$$r = k - \frac{1}{r},$$

which is the assumption employed above. The assumption gives a choice between two values of r, one being the reciprocal of the other. We must clearly choose the value which gives a decreasing not an increasing geometrical progression. r and its reciprocal are the roots of the equation

$$r^2 - kr + 1 = 0.$$

Hence we have

$$2r = k + \sqrt{(k^2-4)}, \quad \frac{2}{r} = k - \sqrt{(k^2-4)}, . \quad . \qquad (15)$$

where

$$k = \frac{\omega^2}{\mu} - 2, \quad \sqrt{(k^2-4)} = \frac{\omega}{\mu}\sqrt{(\omega^2 - 4\mu)}.$$

Thus the resulting motion is completely determined. Successive particles will be opposite in phase, and their amplitudes will diminish in geometrical progression as we move away from the particle which is subjected to external constraint, the diminution being the more rapid as the frequency is greater.

§ 6. The constraining force (taking the mass of the particle as unity) is the excess of the actual acceleration $-\omega^2 y_0$ above the acceleration $-2\mu(y_0 - y_1)$ or $-2\mu y_0\left(1 + \frac{1}{r}\right)$ due to the tensions. It is therefore $y_0\left(-\omega^2 + 2\mu + \frac{2\mu}{r}\right)$.

But, from above,

$$\frac{2\mu}{r} = \omega^2 - 2\mu - \omega\sqrt{(\omega^2 - 4\mu)}.$$

Hence the constraining force is

$$-y_0\omega\,(\omega^2-4\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It is proportional and opposite to the displacement y_0, and therefore does equal amounts of positive and negative work. These remarks apply to the permanent regime only.

The critical value of ω which separates free from forced vibrations is $2\sqrt{\mu}$, and this value makes the constraining force zero. As ω increases from this minimum to infinity, the constraining force increases from zero to infinity, and the ratio r of successive amplitudes increases from unity to infinity.

The work done in the initial stage is represented by the energy of the chain in the permanent state. The ratio of the energy of the whole chain (extending to infinity on both sides) to the energy of the particle to which the constraint is applied is

$$1+2\left(\frac{1}{r^2}+\frac{1}{r^4}+\frac{1}{r^6}+\text{\&c.}\right)=\frac{r^2+1}{r^2-1},$$

which by (15) reduces to

$$\frac{k\{k+\sqrt{(k^2-4)}\}}{\sqrt{(k^2-4)}\,\{k+\sqrt{(k^2-4)}\}}=\frac{k}{\sqrt{(k^2-4)}}=\frac{\omega^2-2\mu}{\sqrt{\{\omega^2(\omega^2-4\mu)\}}}.$$

When ω has its least value $2\sqrt{\mu}$ this is infinite, as it ought to be, since we have in this case an infinite trail of equal oscillations. As ω increases, the value steadily diminishes to the limit unity, showing that the frequency may be so great as practically to confine the energy to the first particle.

§ 7. This discussion of the simple harmonic vibrations of a chain of particles serves to explain Sir George Stokes's illustration of fluorescence, as quoted in Tait's 'Light,' pp. 161–163.

When the frequency of the ætherial vibrations is below the critical value, any nascent disturbance is carried off to a distance by undulations; but when it is above the critical value the effects accumulate at the origin of the disturbance. In the latter case, when the applied force ceases to act, the subsequent motion is compounded of *free* simple harmonic vibrations; and for none of these does the frequency exceed

the critical value. Hence there is a change from higher to lower frequency, and therefore a lessening of refrangibility.

In support of the view that there is a critical frequency for a fluorescent substance, Sir G. Stokes says :—

"In dealing with a single fluorescent substance—not a mixture of two or more—I have generally found that the following feature is (very approximately, at any rate) observed :—As we take incident light of increasing refrangibility, it is at first inactive; then, on reaching a certain point P of the spectrum, it begins to produce fluorescence, and the heterogeneous fluorescent light contains refrangibilities not extending beyond P. As we continue to progress in the incident spectrum, the highest refrangibility of the fluorescent light does not follow the refrangibility of the incident light, but remains about P."

Professor Preston maintains that there is no difference in kind between fluorescence and the process by which lamp-black transforms luminous into non-luminous radiation. In the application of our analogy to lamp-black the critical frequency will be below the range of visibility.

§ 8. The fundamental modes of *stationary* vibration for the chain are most easily deduced from the consideration of two travelling undulations. If the fixed ends coincide with two of the particles, and the intervening length is ma, the equation for a fundamental mode is

$$y = 2A \sin 2\pi x/\lambda \cos 2\pi t/T,$$

λ being either $2ma$ or any submultiple of it exceeding $2a$. This gives $m-1$ modes.

As an example, if the distance between the fixed ends is $4a$, the values of λ are

$$8a, \qquad 4a, \qquad 8a/3.$$

In the first mode the amplitudes of the three free particles are as the sines of

$$\pi/4, \qquad \pi/2, \qquad 3\pi/4.$$

In the second mode, as the sines of

$$\pi/2, \qquad \pi, \qquad 3\pi/2.$$

In the third mode, as the sines of

$$3\pi/4, \qquad 3\pi/2, \qquad 9\pi/4.$$

§ 9. Another mechanical illustration that has been often mentioned as analogous to certain optical phenomena is the mutual influence of pendulums.

First, suppose two simple pendulums of masses 1 and s, with natural frequencies $\omega_1/2\pi$ and $\omega_2/2\pi$, their bobs being at the same level, and elastically connected so that there is a mutual push or pull according as their distance is less or greater than when both pendulums are vertical. The connexions are supposed to be of negligible mass, and the vibrations to be so small that the vertical component of the push or pull is negligible. The movements are supposed to be confined to one vertical plane.

Let x denote displacement of the mass 1 from the vertical through its point of support, reckoned positive when towards s, and ξ the displacement of the mass s, reckoned positive when away from the mass 1. Then $x-\xi$ is their approach, and $\xi-x$ their recess. Let $\mu(x-\xi)$ be the push, and $\mu(\xi-x)$ the pull.

If the masses were unconnected, we should have

$$\ddot{x} = -\omega_1^2 x, \qquad \ddot{\xi} = -\omega_2^2\xi. \quad . \quad . \quad . \quad . \quad (17)$$

When they are connected, we have

$$\left.\begin{aligned} \ddot{x} &= -\omega_1^2 x + \mu(\xi - x), \\ \ddot{\xi} &= -\omega_2^2\xi + \frac{\mu}{s}(x-\xi), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When the system is vibrating in a fundamental mode, x and ξ are in a constant ratio. Assume $\xi = kx$; then equations (18) become

$$\left.\begin{aligned} \ddot{x} &= -\{\omega_1^2 + \mu(1-k)\}x, \\ k\ddot{x} &= -\left\{\omega_2^2 + \frac{\mu}{s}\left(1-\frac{1}{k}\right)\right\}kx, \end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

By division and reduction, we have the quadratic in k,

$$\mu k^2 + k\left(\omega_2^2 - \omega_1^2 + \frac{\mu}{s} - \mu\right) - \frac{\mu}{s} = 0, \quad . \quad . \quad . \quad (20)$$

which has a positive root k_1 and a negative root $-k_2$.

Let the corresponding values of $2\pi/T$ for the fundamental modes be called Ω_1 and Ω_2. For both of them we have,

by equations (19),

$$\Omega^2 = \omega_1^2 + \mu(1-k) = \omega_2^2 + \frac{\mu}{s}\left(1 - \frac{1}{k}\right). \quad . \quad . \quad . \quad (21)$$

Putting $-k_2$ for k, we have

$$\Omega_2^2 = \omega_1^2 + \mu(1+k_2) = \omega_2^2 + \frac{\mu}{s}\left(1 + \frac{1}{k_2}\right), \quad . \quad . \quad (22)$$

showing that Ω_2 is greater than either ω_1 or ω_2; in other words, that the fundamental mode in which the displacements are opposite has a higher frequency than the vibration of either pendulum alone.

Putting k_1 for k, we have

$$\Omega_1^2 = \omega_1^2 + \mu(1-k_1) = \omega_2^2 + \frac{\mu}{s}\left(1 - \frac{1}{k_1}\right). \quad . \quad . \quad . \quad (23)$$

When k_1 is different from unity, one of the two quantities $1-k_1$ and $1-1/k_1$ is positive and the other negative; hence Ω_1 is intermediate between ω_1 and ω_2. Equation (20) shows that, if k_1 is unity, ω_1 is equal to ω_2; that is, the pendulums are of equal length. Whenever they are unequal, the fundamental mode in which their displacements are similar is intermediate in frequency between the vibrations of the two pendulums singly.

The actual vibration of the system will be either that corresponding to k_1 and Ω_1, or that corresponding to $-k_2$ and Ω_2, or a combination of the two in an arbitrary ratio; according to the initial circumstances.

If the pendulums are nearly equal, both in length and mass, the coefficient $\omega_2^2 - \omega_1^2 + \mu/s - \mu$ of k in the quadratic is small and the positive and negative roots are nearly equal, also their product $-1/s$ is -1 nearly; hence the roots are approximately ± 1. The values of Ω^2 from (21) are therefore approximately

$$\Omega_1^2 = \omega_1^2, \quad \Omega_2^2 = \omega_1^2 + 2\mu,$$

the latter being always the greater, as proved above.

In practice μ is usually exceedingly small; hence Ω_1 and Ω_2 are nearly equal. The general equations

$$\left.\begin{aligned} x &= \mathrm{A}\cos(\Omega_1 t - \alpha_1) + \mathrm{B}\cos(\Omega_2 t - \alpha_2), \\ \xi &= k_1 \mathrm{A}\cos(\Omega_1 t - \alpha_1) - k_2 \mathrm{B}\cos(\Omega_2 t - \alpha_2), \end{aligned}\right\} \quad . \quad (24)$$

will accordingly represent what are called in acoustics "curves of beats," the beats being strongest when A and B (and therefore also k_1A and k_2B) are nearly equal.

If we start from a condition in which the pendulum x is at rest in the zero position, the equations will be

$$\left.\begin{aligned} x &= A(\cos\Omega_1 t - \cos\Omega_2 t), \\ \xi &= A(k_1\cos\Omega_1 t + k_2\cos\Omega_2 t) \\ &= A(\cos\Omega_1 t + \cos\Omega_2 t), \text{ nearly,} \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

When t is such that $\cos\Omega_1 t$ and $\cos\Omega_2 t$ are each sensibly unity, the first pendulum will have sensibly zero excursions, and the second pendulum will have maximum excursions. When one of the cosines is sensibly 1, and the other sensibly —1, these conditions will be reversed. In fact, we have

$$\left.\begin{aligned} x &= 2A\sin\frac{\Omega_2-\Omega_1}{2}t \, . \, \sin\frac{\Omega_2+\Omega_1}{2}t, \text{ rigorously,} \\ \xi &= 2A\cos\frac{\Omega_2-\Omega_1}{2}t \, . \, \cos\frac{\Omega_2+\Omega_1}{2}t, \text{ nearly,} \end{aligned}\right\} \quad . \quad (26)$$

showing that the excursions are

$$2A\sin\tfrac{1}{2}(\Omega_2-\Omega_1)t \quad \text{and} \quad 2A\cos\tfrac{1}{2}(\Omega_2-\Omega_1)t,$$

each of which in its turn vanishes when the other is 2A. This can be illustrated by hanging two equal pendulums from the same stand. As regards phases, the comparison of the two factors

$$\sin\tfrac{1}{2}(\Omega_2+\Omega_1)t \quad \text{and} \quad \cos\tfrac{1}{2}(\Omega_2+\Omega_1)t$$

shows that, at first, ξ is earlier than x by a quarter period.

§ 10. Next take the case of one pendulum suspended from another, each consisting of a heavy particle at the end of a weightless thread.

Upper mass	1,	Lower mass	s ;
Length of upper pendulum	a,	of lower	b ;
Displacement of upper mass	x,	of lower	ξ ;

the displacements being measured horizontally from a vertical through the fixed point of support, and being so small that vertical accelerations may be neglected. Then, since the

tensions are sg and $(1+s)g$, we have

$$\left.\begin{aligned} s\ddot{\xi} &= -sg\frac{\xi-x}{b}, \qquad \ddot{\xi}=-\frac{g}{b}(\xi-x), \\ \ddot{x} &= sg\frac{\xi-x}{b}-(1+s)g\frac{x}{a} = \frac{sg}{b}\xi-g\left(\frac{s}{b}+\frac{1+s}{a}\right)x. \end{aligned}\right\} \quad . \quad (27)$$

Assume $\xi=kx$ for a fundamental mode. Then the equations become

$$\left.\begin{aligned} k\ddot{x} &= -\frac{g}{b}(k-1)x \\ \ddot{x} &= -\frac{g}{b}\left\{s+\frac{b}{a}(1+s)-ks\right\}x \end{aligned}\right\} \quad . \quad . \quad (28)$$

By division and reduction, we obtain the quadratic for k

$$ask^2-k\{b-a+(b+a)s\}-a=0\,; \quad . \quad . \quad . \quad (29)$$

which, as in the previous example, has a positive root k_1 and a negative root $-k_2$

As before, denoting the acceleration-factor $-\ddot{x}/x$ or $(2\pi/\mathrm{T})^2$ by Ω^2, the two equations give

$$\Omega^2=\frac{g}{b}\left(1-\frac{1}{k}\right)=\frac{g}{a}\left\{1+s+\frac{as}{b}(1-k)\right\}. \quad . \quad . \quad (30)$$

Putting $-k_2$ for k all the terms are positive; hence the first expression for $\Omega_2{}^2$ is greater than g/b, and the second is greater than g/a; that is to say:—*The mode in which the displacements are opposite is quicker than either pendulum alone.*

For the other mode, we have

$$\Omega_1{}^2=\frac{g}{b}\left(1-\frac{1}{k_1}\right)=\frac{g}{a}\left\{1+s+\frac{as}{b}(1-k_1)\right\}. \quad . \quad . \quad (31)$$

The first value shows that $\Omega_1{}^2$ is less than g/b. The second value shows that it will be less than g/a if $s+(1-k_1)\,as/b$ is negative, that is if k_1 is greater than $1+b/a$; and this condition is always fulfilled, for the substitution of this value of k in the quadratic gives an opposite sign to the substitution of a very large positive quantity. Hence *the mode in which the displacements are similar is slower than either pendulum alone.*

The physical meaning of the result $k_1>1+b/a$ is that, in

the mode in which the displacements are similar, the lower string is more inclined to the vertical than the upper.

When a is infinite, $-k_2$ is $-1/s$, and $\Omega_2{}^2$ is $(1+s)g/b$.

When b is infinite, k is zero, and Ω^2 is $(1+s)g/a$.

In the former case, the lower string rotates about the common centre of gravity as a fixed point, so that the virtual length of the pendulum is $b/(1+s)$. In the latter, the lower string simply alters the downward force on the upper mass from g to $(1+s)g$.

§ 11. When the ratio s of the lower to the upper mass is very small, and a and b are not approximately equal, the quadratic (29) may be written

$$k^2+k\frac{a-b}{as}-\frac{1}{s}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Neglecting the first term (which is small compared with the others), we have

$$k=\frac{a}{a-b},$$

as the approximate value of one root. It is the value obtained by putting $s=0$ in the general quadratic, and is identical with the value of k obtained by supposing the point of support constrained to vibrate like a pendulum of length a. The other approximate root is

$$k=\frac{b-a}{as};$$

for the product of the two roots must be $-1/s$.

It will be noted that the first root makes the excursions of the two pendulums comparable with one another; whereas the second root makes the excursions of the upper very small compared with the lower.

The closest approximations to the values of Ω^2 are got by employing in connexion with the first root the formula

$$\Omega^2=\frac{g}{a}\left\{1+s+\frac{as}{b}(1-k)\right\},$$

and in connexion with the second root the formula

$$\Omega^2=\frac{g}{b}\,1-\frac{1}{k}.$$

The specifications of the two modes will accordingly be

$$\left.\begin{array}{ll} \text{First Mode.} & \text{Second Mode.} \\ k=\dfrac{a}{a-b}, & k=\dfrac{b-a}{sa}, \\ \Omega^2=\dfrac{g}{a}\left(1+\dfrac{sb}{b-a}\right). & \Omega^2=\dfrac{g}{b}\left(1+\dfrac{sa}{a-b}\right). \end{array}\right\} \quad . \quad . \quad . \quad (33)$$

The first mode nearly agrees in period with a pendulum of length a, and the second with a pendulum of length b.

The two values of k are obviously opposite in sign. The positive k always makes Ω^2 less than the lesser of g/a and g/b, and the negative k makes it greater than the greater, in accordance with the general rule. When a is infinite one value of Ω^2 is $(1+s)g/b$, and when b is infinite one value is $(1+s)g/a$, as previously found for the general case.

§ 12. The results obtained in the preceding section agree with the statements contained in Lord Kelvin's paper* "On the Rate of a Clock or Chronometer as influenced by the Mode of Suspension," some of which are quoted in recent editions of Tait and Steele's 'Dynamics.'

Lord Kelvin selects the second mode as the practical mode for a pendulum supported on a yielding stand, presumably because the amplitude of the pendulum in the first mode is comparable with that of the point of support and therefore inappreciable. He also selects it as the practical mode for an ordinary spring clock suspended like a compound pendulum from a fixed axis; presumably because the escapement would not work with such an enormous departure from correct time as the first mode would involve.

On the other hand, Lord Rayleigh, in a paragraph which is often quoted in discussions on anomalous dispersion, ignores the second mode and adopts the first. The justification seems to be that his argument relates to the behaviour of the upper pendulum, and that in the second mode the excursions of the upper pendulum are infinitesimal.

In the first mode, the ratio of the amplitude of the upper to that of the lower pendulum is $1/k=(a-b)/a$, which may be small, but not negligible, and the ratio of the changed

* Glasgow, Trans. Inst. Engin. x. 1867, pp. 139–150.

to the natural (frequency)^2 for the upper is $1+sb/(b-a)$. The upper pendulum is therefore quickened if $b-a$ is positive, that is, if the lower pendulum is naturally the slower, and is retarded if the lower pendulum is naturally the quicker. Lord Rayleigh suggested many years ago that the paradoxical appearances presented in "anomalous dispersion" are due to an action analogous to this. In anomalous dispersion there is always excessively strong selective absorption, evidenced by a black band in the spectrum produced by a prism of the anomalous substance; and the colours which are not thus blotted out are displaced from their usual order. A general statement of the facts, which has been deduced from the comparison of a number of anomalous spectra, is that, in passing through the colours of the ordinary spectrum from red to violet, the refractive index is increased where the absorption increases rapidly, and diminished where absorption diminishes rapidly. In the case of iodine vapour, red is thus rendered more refrangible than blue and violet.

According to Lord Rayleigh, we are to regard the vibrating æther as analogous to the heavy upper pendulum, and the vibrating molecules of the substance to the light lower pendulum. The colour which is absorbed is the colour which has the same frequency as the molecules of the substance. Colours which, in the ordinary spectrum, are on the red side of the absorbed colour have lower frequency than the molecular vibrations. This is the case of "upper pendulum naturally the slower," and the resultant vibration will be slower still. Lord Rayleigh calls this effect an increase in the "virtual inertia" of the upper pendulum. I subjoin Lord Rayleigh's own words (Phil Mag. xliii. p. 322), which are quoted in full in Lord Kelvin's 'Baltimore Lectures,' and are paraphrased in Preston's 'Light.' Professor Preston identifies the "pendulum" in the quotation with the lower of the two, and the "point subject to horizontal vibration" with the bob of the upper; and this is also my own understanding of the passage.

"The effect of a pendulum, suspended from a point subject to horizontal vibration, is to increase or diminish the virtual inertia of the mass, according as the natural period of the

pendulum is shorter or longer than that of its point of suspension. This may be expressed by saying that, if the point of support tends to vibrate more rapidly than the pendulum, it is made to go faster still, and *vice versâ*.

"Below the absorption band, the material vibration is naturally higher, and hence the effect of the associated matter is to increase abnormally the virtual inertia of the æther, and therefore the refrangibility. On the other side the effect is the reverse."

The latter part of the passage is to me somewhat obscure. The analogy seems to point to a change of frequency, but instead of this we have a change in velocity of propagation.

§ 13. When $a-b$ approaches zero, the foregoing approximation is insufficient, and the following investigation is preferable.

If the coefficient of k in the quadratic (29) vanishes, we have

$$\left.\begin{aligned} s&=\frac{a-b}{a+b}, & k&=\pm\frac{1}{\sqrt{s}}, \\ \Omega_1^{\,2}&=\frac{g}{b}(1-\sqrt{s}), & \Omega_2^{\,2}&=\frac{g}{b}(1+\sqrt{s}). \end{aligned}\right\} \quad . \quad . \quad . \quad (34)$$

As s, being the ratio of the masses, cannot be negative, these conditions require $a-b$ to be positive, and s will lie between the limits 0 and 1. We shall have

$$\left.\begin{aligned} x&=\mathrm{A}\cos\Omega_1 t+\mathrm{B}\cos(\Omega_2 t-\alpha), \\ \xi&=\frac{1}{\sqrt{s}}\left\{\mathrm{A}\cos\Omega_1 t-\mathrm{B}\cos(\Omega_2 t-\alpha)\right\}; \end{aligned}\right\} \quad . \quad . \quad . \quad (35)$$

and if the initial values of x, ξ, and $\dot{x}$ are zero, we have $\alpha=0$, $\mathrm{A}=-\mathrm{B}$; hence

$$\left.\begin{aligned} x&=\mathrm{A}\,(\cos\Omega_1 t-\cos\Omega_2 t), \\ \xi&=\frac{\mathrm{A}}{\sqrt{s}}\,(\cos\Omega_1 t+\cos\Omega_2 t). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If s is small, Ω_1 and Ω_2 are nearly equal, and each pendulum oscillates with amplitude varying between zero and a maximum, the maximum of one coinciding with the zero of the other. The maximum amplitude is $2\mathrm{A}$ for x, and $2\mathrm{A}/\sqrt{s}$

for ξ. The phases are, at first, a quarter-period earlier for ξ than for x. (See equations (26).)

If the initial conditions are altered by making ξ vanish instead of x, the plus and minus in the expressions for x and ξ will be interchanged, and the phases will, at first, be a quarter-period earlier for x than for ξ, the equations being now reducible to the forms

$$\left.\begin{aligned} x &= 2\mathrm{A} \cos \tfrac{1}{2}(\Omega_2 - \Omega_1)t \,.\, \cos \tfrac{1}{2}(\Omega_2 + \Omega_1)t, \\ \xi &= \frac{2\mathrm{A}}{\sqrt{s}} \sin \tfrac{1}{2}(\Omega_2 - \Omega_1)t \,.\, \sin \tfrac{1}{2}(\Omega_2 + \Omega_1)t. \end{aligned}\right\} \quad . \quad . \quad (37)$$

This investigation applies to the experiment described in Rayleigh on Sound, 2nd edition § 62 ; but the experiment appears to have been stopped as soon as the lower pendulum attained its first maximum.

§ 14. Sellmeier (Pogg. *Ann.* vol. cxlv. p. 534) refers to the transference of energy which goes on from one pendulum to the other when the amplitudes vanish alternately, and maintains that a similar transference of energy between particles embedded in the æther and particles of a fluorescent body is the cause of fluorescence. He works out the case in which s is small, $a=b$, and one of the pendulums is initially at rest in the zero position.

XIX. *On the Circulation of the Residual Gaseous Matter in a Crookes Tube. By* ALAN A. CAMPBELL SWINTON*.

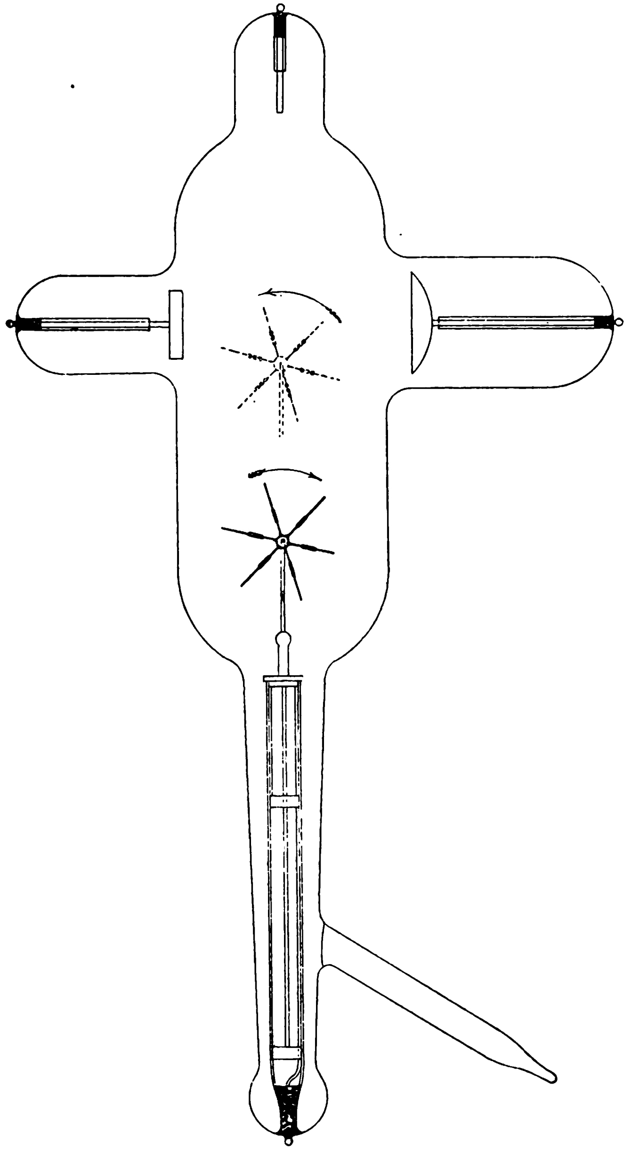
THERE appears to be generally some doubt as to the way in which the particles or atoms that form the cathode-rays in a Crookes tube return again to the cathode. It is obvious that they must return, as otherwise they would all become collected at one end of the tube, and the cathode would soon be surrounded by an absolute vacuum. By some it has been supposed that the atoms return by ordinary diffusion in the intervals between the succeeding electrical discharges; by some that they creep back along the inner surface of the glass walls of the tube; by others that they return during a discharge through the space between the cathode-rays and the glass. Further, there is the question whether the returning atoms carry any portion of the positive electricity from the anode to the cathode, similarly as part at any rate of the negative electricity is believed to be carried from the cathode to the anodic portion of the tube by the cathode-ray atoms.

In his 1891 Presidential Address to the Institution of Electrical Engineers, Sir William Crookes described a tube which was divided into two halves by a diaphragm pierced with two small apertures. Near each aperture was mounted a small wheel with vanes to detect and show the direction of any stream of atoms that might pass through. It was found that when the cathode was caused to project rays through one aperture, the rotation of the wheel at the other aperture showed the atoms in the act of returning.

In this tube, however, both anode and cathode were on the same side of the diaphragm, the anode being behind the cathode. So the experiment cannot be said to decide the existence of a true anode-stream, but merely to demonstrate that the action of the cathode-stream was to create a difference of gaseous pressure in the two halves of the tube which relieved itself by a current of atoms through the spare aperture in the diaphragm.

* Read March 25, 1898.

Fig. 1.



The writer, with the assistance of Mr. J. C. M. Stanton and Mr. H. Tyson Wolff, has investigated the matter further by means of a series of tubes, one of which is illustrated in

fig. 1. In this tube, which is very highly exhausted, we have as two electrodes a concave aluminium cup and an aluminium plate placed opposite to one another, as in an ordinary focus-tube. As will be seen, there is also a supplemental wire electrode at one side. Inside the tube is a very delicately pivoted radiometer-wheel with mica vanes, which is so mounted on a sliding-rod that by simply shaking the tube the wheel can be moved out into the centre, as indicated by the dotted lines, so that the cathode-stream impinges directly upon the vanes, or can be moved back to the position shown in full lines in the illustration, when the vanes are quite out of the cathode-stream. When the wheel is in the former position the tube acts exactly as an ordinary Crookes electric radiometer-tube, the wheel rotating one way or another in the direction of the cathode discharge as the concave cup or the flat-plate electrode is made cathode, the rotation being much more rapid in the latter than in the former case.

On the other hand, if the wheel is moved well out of the cathode-stream, provided the exhaustion is high enough, it is found to rotate in the opposite direction to the cathode-rays, that is to say, in the direction that indicates an atomic stream from anode to cathode round the outside of the cathode-stream. The rotation will only take place with high exhaustion, and is never so rapid as in the previous case when the vanes were in the cathode-stream; but it is faster and faster the higher the exhaustion, and at very high vacua the speed is very considerable. The wheel rotates whether the concave or the flat plate is made cathode, but in either case in a direction opposite to what it did when in the cathode-stream.

Very little electric power is necessary to show this effect. With a sufficiently high vacuum a small Wimshurst machine, passing so little current through the tube that scarcely any fluorescence is visible, will cause the wheel to rotate at a speed of many turns per second; and rotation can even be produced by the mere approach of two oppositely charged leyden-jars to the terminals of the tube.

Further experiment appears to show that the rotation is really due to a stream returning round about the cathode-stream from anode to cathode, and that the atoms or particles

of which this stream consists are positively charged. In the experiments so far described the anode was also the anti-cathode; and it is obvious that the stream might not be truly an anode-stream, but merely a reflected or splashed cathode-stream. It is, however, found that if the flat plate used as anode be earthed, or if the plate be disconnected from the electrical source of power, and used merely as an anti-cathode, the spare electrode already alluded to opposite to the wheel being employed as anode, the wheel refuses to rotate, though rotation recommences immediately in the first case when the earth-connexion is removed, and in the latter case when the anode and anti-cathode are connected together. This appears to be fairly conclusive evidence that it is a stream projected from the anode that causes the rotation of the wheel. It might, however, be argued that this stream, though coming from the anode, is in reality a cathode-stream due to oscillations in the electric discharge causing the anode to be at times negatively charged. Against this view it should be stated that the effects are produced quite as well by the silent discharge from a small Wimshurst machine as with Ruhmkorff-coil discharges. It has further been noted that the wheel rotates in the proper direction with either the flat plate or the concave cup employed as cathode, the other being used as anode; and any cathode-rays given off by the concave cup would be concentrated, and could not therefore reach the wheel in the position in which it is used to show the stream from the anode. Again, it might be supposed that the rotation is due to heat-radiation from the anti-cathode. In this case, however, the rotation should occur when the side anode is employed, which, as mentioned, is not the case. Further, even when such powerful discharges are used as to make the anti-cathode visibly red hot, the rotation of the wheel ceases almost immediately the current is interrupted, the movement that continues being obviously due to the momentum of the wheel.

The experiments therefore appear to establish the existence at high exhaustions of a true anode-stream which travels from the anode to the cathode, just as does the cathode-stream from the cathode to the anti-cathode, the anode-stream passing round the exterior of the cathode-stream at a considerably

lower velocity than the latter, but at a greater and greater velocity the higher the exhaustion.

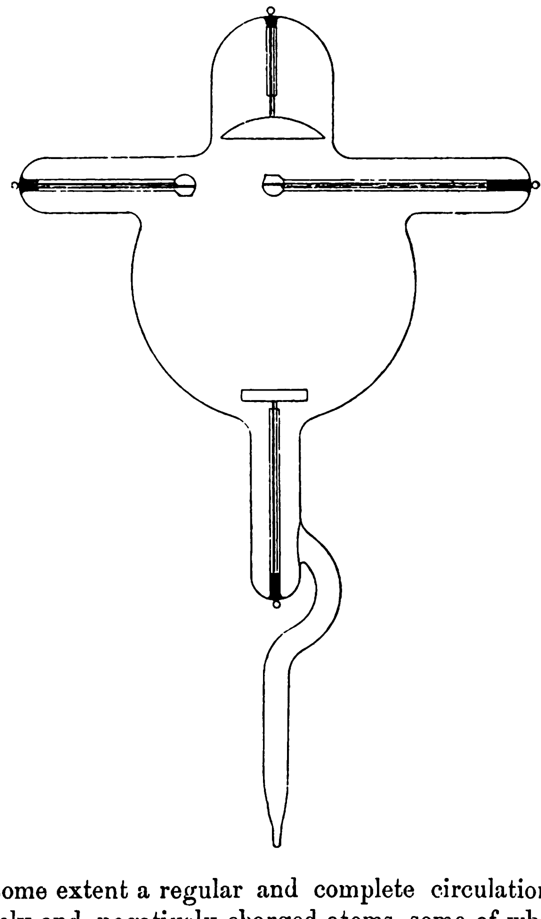
It also appears that while, as is well known, the cathode-stream is negatively charged, the anode-stream is charged positively.

For the purpose of ascertaining this a tube fitted with exploring-poles, as used by Sir William Crookes, was employed. These poles were, however, somewhat differently arranged to any described in Sir W. Crookes's papers, while the exhaustion, which was taken to the degree required for Röntgen-ray work in which the negative dark space appears to fill the whole tube, was probably much higher than that Sir W. Crookes employed. Such a tube is illustrated in fig. 2, and contains the usual aluminium cathode-cup and anode-plate. The exploring-poles consist of aluminium wires tipped with platinum plates enclosed in glass tubes, which are blown out into small bulbs at the free extremities so as to contain and shield the platinum plates, the bulbs having, however, each an aperture exposing the platinum on one side. One platinum plate is arranged just opposite to the centre of the cathode-cup, and the aperture in its containing glass bulb faces the cathode so that the cathode-rays can impinge upon the platinum. The other and shorter pole has its platinum plate well to one side of the cathode, and has the aperture in its glass cup turned away from the cathode towards the anode. Experiments were conducted with this tube highly exhausted and excited by means of an induction-coil, the polarity of the two exploring-poles being ascertained by means of a quadrant-electrometer. With the aluminium cup as cathode and the plate as anode, the longer exploring-pole, which has its bare extremity facing the cathode-stream, was found invariably to be charged negatively, while the other and shorter exploring-pole was found always to be charged positively. This was found to be more and more distinctly the case the higher the exhaustion.

It seems therefore that at high vacua at any rate some portion of the positive electricity passing through the tube is carried by the positively charged atoms or particles that form the anode-stream. Very probably at lower exhaustions the electric discharge passes through the tube chiefly by an inter-

change of electrical charges from molecule to molecule on the Grothüss chain principle. At very high exhaustions, however, when the mean free path becomes considerable, this may cease to be the case, at any rate to a large degree; and there may

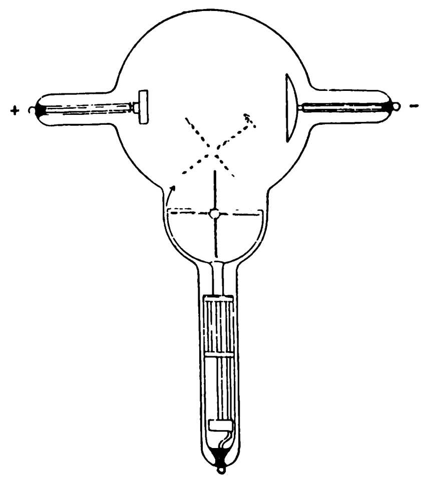
Fig. 2.



be to some extent a regular and complete circulation of the positively and negatively charged atoms, some of which may make the entire journey from anode to cathode, or *vice versâ*, and deliver up their charges not by interchange with other gaseous atoms, but by direct convection to the electrodes of opposite sign.

It may be mentioned that for showing the movements of the streams a tube of the form illustrated in fig. 1 is not essential, the anode-stream being equally well marked in a tube of the ordinary globular form, provided the wheel is mounted so as to be half contained within a glass cup arranged so as to prevent the stream acting equally and oppositely upon the vanes upon diametrically opposite sides of the wheel, as shown in fig. 3. Further, it is not necessary to employ the

Fig. 3.



sliding adjustment for the wheel, as the effects can be shown equally well by means of two separate tubes, one with its wheel in the forward position and the other with its wheel in the back position. The two tubes can be operated simultaneously by connecting them in series.

It should also be stated that for these experiments extremely high vacua are requisite, and that with a Ruhmkorff coil as the source of electrical power the effects can only be shown satisfactorily with the tube connected to the mercury-pump, for the reason that the discharges from the coil inevitably bring down the vacuum very quickly, apparently by their action upon the mica vanes, which are visibly affected when the cathode-stream from the concave cup is allowed to fall upon them. Using a small Wimshurst machine, however,

the effects can be shown after the tube has been sealed off, though even then with use the vacuum appears to deteriorate in a short time.

Discussion.

Prof. Boys said he did not feel altogether convinced by the experiments that the rotation of the mill was due to simple mechanical motion of the particles of matter between the electrodes. The weight of air left in the tube at such high degrees of exhaustion was extremely small; it was difficult to realize that its impact could produce the sudden mechanical effect observed at the moment of the reversal of the rotation of the mill.

Mr. Wimshurst thought it important to keep in mind the existence of mercury vapour in the tube. He also referred to some experiments in which a bar of metal was used to explore a focus tube, by observations of the changes of luminosity produced in different positions.

Dr. Chree said that if the rotations of the mill could be shown to indicate a velocity of the particles, of the same order as that observed in Crookes's experiments, it was safe to assume the existence of a similar cause. This might be important in deciding as to the general truth of the bombardment theory of Crookes. He asked whether the rotation had been investigated within the dark space around the cathode.

Mr. Appleyard suggested that in tracing the cause of the rotation, it would lead to simpler results if the vanes of the mill were made of some light conducting substance. Mica introduced difficulties, owing to its retention of the charges.

Prof. Boys pointed out that this could be done by gilding the mica.

Mr. Campbell Swinton, in reply, said that the objection raised by Prof. Boys to the mechanical theory of the rotation would apply equally to the whole theory of electro-radiometry, including the case of the mill used originally by Crookes in the direct path of the cathode stream. But it must be remembered that, although the mass of matter present within the tube was very small, its velocity was

proportionally great; it was of the order of 9000 kilometres per second. Hence, the contained matter might be conceived as capable of producing the observed acceleration, and Crookes's bombardment theory might safely be adopted as a good working hypothesis. In the tubes used for these experiments, the exhaustion was carried so high that the negative dark space appeared to fill the whole tube. He had, so far, only tried mica for the vanes, but he thought it would be important to observe the results with a substance that did not retain the charges.

XX. *Some Further Experiments on the Circulation of the Residual Gaseous Matter in Crookes Tubes.* By ALAN A. CAMPBELL SWINTON*.

IN the discussion which followed my former paper on this subject some objection was taken to the use of a non-conducting substance, *i. e.* mica, for the vanes of the mill which was used to detect the circulation of the ultra-gaseous matter, it being suggested by Professor Boys that the rotation produced might be the result of electrification of the vanes. It was further suggested by Mr. Appleyard that gilding the vanes, so as to make them conductive, might modify the effect.

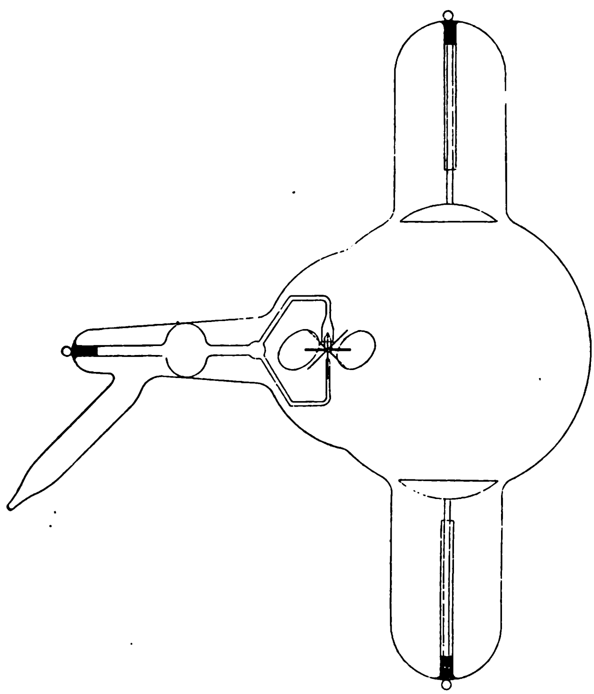
Mr. Wolff has now constructed for me a tube similar to fig. 1 of my former paper, but with the mica vanes gilded and mounted on a brass cap revolving upon a steel needle-point connected with a wire and terminal, so that the vanes can readily be earthed.

In this condition, and with the vanes so placed as to be outside the cathode-stream, it is found that this wheel behaves in a similar manner to the former non-conducting and insulated wheel. It shows a greater tendency to assume a position of stability, due evidently to electrostatic induction; but though this renders it sometimes rather troublesome in starting, still when once under weigh it will continue to rotate as long

* Read May 27, 1898.

as the tube is excited. It will occasionally, when starting, make a few reverse revolutions, due probably to electrostatic influence and momentum, and also possibly to eddy-currents in the residual gaseous matter; but it is found that when it does this it invariably reverses its rotation almost immediately, and proceeds to rotate more and more rapidly in the direction that indicates a stream of residual gaseous matter passing from the anode to the cathode. Usually, after one or two oscillations, it starts immediately to rotate in this direction.

An electrometer connected to the wheel through the pivot and needle-point shows that the vanes are always positively electrified.



In order further to investigate the matter, I have had constructed another form of radiometer-tube, as shown in the accompanying illustration. Here the vanes (of mica) are inclined, and the axis is parallel to the line joining the cathode and anode. The wheel thus rotates in a plane at right angles

to the cathode- and anode-streams; and it is difficult to see how electrification of the vanes should in any way assist its rotation either in one direction or in another.

The wheel is arranged so that the vanes are all outside the cathode-stream, and when the tube is excited it is found invariably to rotate in a direction indicating a stream from anode to cathode. Concave aluminium cups are used for both electrodes, and the direction of rotation of the wheel is found immediately to reverse when the positive and negative connexions are transposed.

These experiments consequently confirm the hypothesis suggested in my former paper, that at very high exhaustions there exists a molecular or atomic stream from anode to cathode which carries a positive charge and travels at considerable velocity outside of the opposite cathode-stream.

Discussion.

Mr. J. Quick asked what was the minimum degree of exhaustion required to produce these results.

Prof. Boys said that the experiment gave some amount of probability to the truth of Mr. Campbell Swinton's hypothesis, but it did not altogether prove the mechanical theory of rotation to be correct. He was glad that a chance suggestion at the last discussion has led to such interesting experiments being continued.

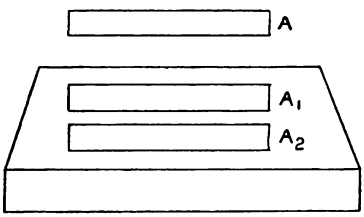
Prof. Threlfall mentioned that Boettger had devised a method for gilding mica, by a chemical process, that was much to be preferred to ordinary gilding.

Mr. Campbell Swinton said it was necessary to exhaust the tubes as completely as possible; to a point where it was only just possible for any discharge at all to pass through them. If the rotation was due to electrification, there must still be some mechanical process whereby the charges get to the vanes—a stream of residual gas satisfied that condition.

XXI. *On a Method of viewing Newton's Rings.* *By* T. C. PORTER*.

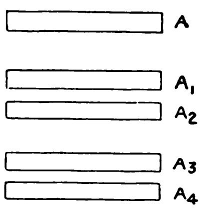
IF rays of light (here supposed parallel to each other) pass through a rectangular slit A (fig. 1) and fall upon a piece of plate-glass of the same thickness as the width of the slit or greater, and if we observe the reflexion of the slit, it appears thus :—

Fig. 1.



A_1, A_2 being the first reflexions of A in the upper and lower surfaces of the glass. If the glass plate be viewed more obliquely, other reflexions, which for the present we shall neglect, will appear, all of them lying below A. If a second glass plate be added below the first, but separated from it by an interval, two more images of A will be seen below A_1, A_2, caused by the reflexion of A in the upper and lower surfaces of this second plate, A_3, A_4 in fig. 2. If now the lower glass plate be moved up till its upper surface is in contact with the

Fig. 2.

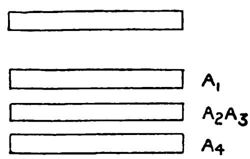


lower surface of the upper plate, the two reflexions A_3, A_4 will be seen to move up with it till, when the two plates are in contact, A_3 coincides with A_2, and the appearance of the

* Read April 22, 1898.

images is that represented in fig. 3. It is evident that the middle image is caused solely by the light reflected from the lower surface of the upper plate, and from the upper surface of the lower plate. If we substitute for the two glass plates the apparatus generally used for exhibiting Newton's rings, we can in this simple way view the rings by light coming from the two interior surfaces only, and thus completely free

Fig. 3.

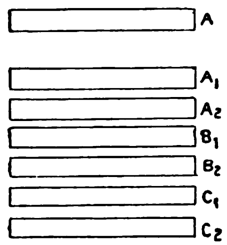


from either light reflected from the upper surface of the upper plate, or from the lower surface of the lower plate.

Thus viewed, the central area appears of a velvety black and the colours of the rings exceedingly brilliant. The whole experiment can be easily projected, and the difference in the appearance of the rings on the screen with and without the slit is very striking. But the interest of the method does not end here; for besides affording an easy and obvious proof that the rings are caused by two reflexions, one at each of the two inner surfaces, and, *under these circumstances*, by these two reflexions *only*, it also supplies a method of seeing and distinguishing the interference-curves caused by light which has undergone 1, 2, or 3 reflexions (forming the ring-system usually seen, and named after Sir Isaac), and the curves formed by the interference of rays which have suffered 4, 5, and 6 reflexions or more. For if the reflexion of the slit in a single glass plate be viewed more obliquely, as suggested before, the images are arranged as in fig. 4, where B_1 is caused by light which, originally reflected from the lower surface of the plate, has undergone a second reflexion from the upper interior surface, and a third from the lower interior surface of the plate. Similarly, the light by which B is seen has undergone five internal reflexions; C has undergone seven; C_1 nine; and so on. Now when the second plate is added beneath the first it gives rise to a similar series, but more complicated

from the fact that the second plate's reflexions are not only caused by its internal surfaces but also by its external upper surface.

Fig. 4.



The course of the rays forming the first few reflexions is easily seen from fig. 5. By backing the plate in the usual way we can practically suppress the reflexions from the lower internal surface of the lower plate; and since A_1 is the sole reflexion from the upper surface of the upper plate, it follows

Fig. 5.

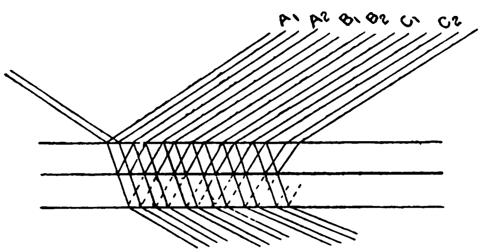
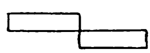


Fig. 5 *a*.



that we see in A_2, B_1, B_2, &c., the results of reflexions in (*a*) the internal upper surface of the upper plate, (*b*) the internal lower surface of the upper plate, (*c*) the external upper surface of the lower plate, and possibly (*d*) the external lower surface of the upper plate (*vide* fig. 8).

Of (*a*), (*b*), and (*c*), the images A_2, B_1, B_2, C_1, C_2 contain rays which have undergone reflexion at these respective sur-

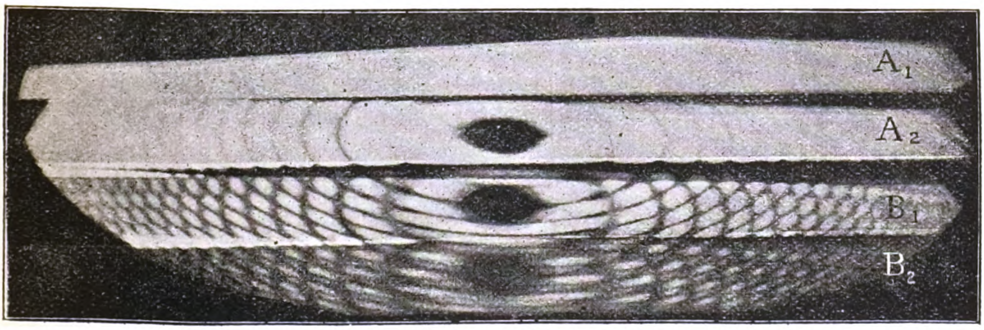
faces the number of times under the corresponding letter in the subjoined table.

	(*a*)	(*b*)	(*c*)
A_2	0	1	1
B_1	1	2	2
B_2	2	3	3
C_1	3	4	4
C_2	4	5	5

Now since the phase of a light-wave loses half a wave-length in the act of being reflected in a denser medium, and since the sum of the three rows is odd for each strip such as B_1, the light which forms the figures seen in each image of the slit will in every case lose an integral number of wave-lengths: it follows that if the centre of the primary rings is black or coloured, so far as this consideration is concerned, the centres of the secondary and tertiary rings will also be black or coloured. B_1, B_2, C_1, C_2 .... will therefore each contain a reflexion of the primary rings, growing weaker and weaker in intensity, not only from the loss of light at each reflexion but also from absorption, and perhaps from the scattering of a very small fraction of the light by solid particles of dust or air-bubbles, which are wont to occur even in the clearest glass. The effect of the curvature of the lower surface of the upper plate will be to displace these repetitions of the primary rings a little downwards. It is clear that any rays which once completely interfere are cut out once for all; and it follows that any interference-curves which appear for the first time in any particular image of the slit must be the result of light which has hitherto escaped interference: *e. g.* if we look at the image B_1, besides the reflexion of the primaries there is a series of rings which are exact continuations of the primaries; and, moreover, *these continuations can be traced*, though the observation is not an easy one, *right across the reflexion of the black central spot which occupies the centre of the primary rings.* If this be not due to light scattered in the body of the glass itself, it must prove that the interference which causes the black spot at the primaries is not complete, though very nearly so. In making this observation the eye must be screened from all light except that which comes from the black spot in B; and it must be borne in mind that

care is necessary to avoid the smallest particles of dust between the two plates. If there are any of these, they cause white specks in the black area, *and in its reflexion in* B, and will obviously make it easy for the eye to follow the black continuations of the primaries across the reflexion of the spot. The writer is of opinion that the interference is not complete. It seems worth notice that when the plates are very clean the darkest area of the black spot has a *sharply defined* edge, recalling the character of the black film of a soap-bubble. If there is a species of welding together of the molecules of glass at the black spot, with molecules of the gases of air mechanically entangled, the blackness would be explained, since the light can pass through without leaving the glass medium; and, on the other hand, the entangled air being, as it were, pulverized, would reflect irregularly. The fact that the edge of the spot is very sharply defined shows that it is not only the effect of the difference of wave-length. As the black spot is approached in the plane of the rings, the light *does* fall off gradually up to a certain point, at which the shade is a dark grey, but then passes "*per saltum*," to the velvety black before alluded to.

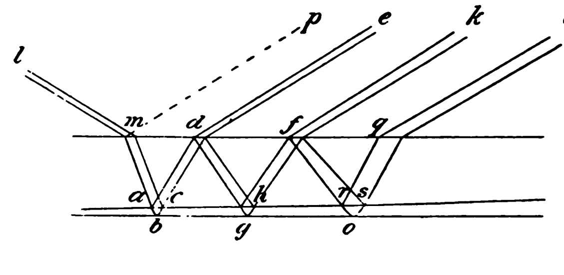
Fig. 6.



To explain how it is the continuations of the primaries are seen in B (*vide* fig. 6), see fig. 7, where lm is one of the rays of white light coming through the slit, reflected at the uppermost surface at m, giving the image A, then transmitted through the upper plate to a and accompanied by light reflected from the upper surface of the lower plate, which

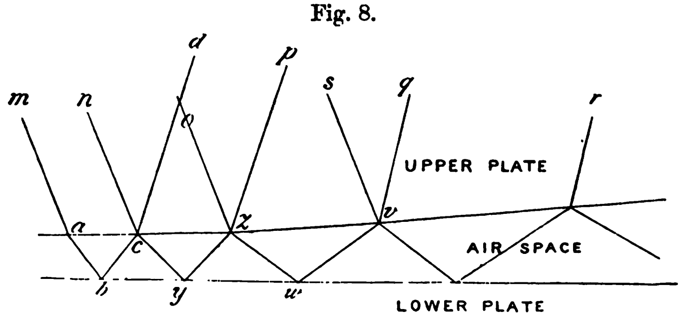
has not interfered with it. At *a* a part of the ray is reflected; the rest is refracted into the air-space, undergoes reflexion at *b*, another refraction at *c*; and is then transmitted to *d*

Fig. 7.



unless destroyed by interference: there, part is refracted into the air and forms one of the bright primaries; but a considerable part is reflected to *h* and *g*, the two reflected rays there interfering for the first time, and generating the continuations of the primary rings; for it is evident that light reflected from *d* might, so far as its effect at *k* is concerned, have been originally incident at *d*, in which case it certainly would have generated the continuations of the parts of the primary rings observed in A. Thus B will show (1) a reflexion of the parts of the primaries seen in A, and (2) continuations of the primaries. It is now perfectly obvious that the image B_2 (fig. 5) ought to exhibit:—(1) Faint continuations of the

Fig. 8.

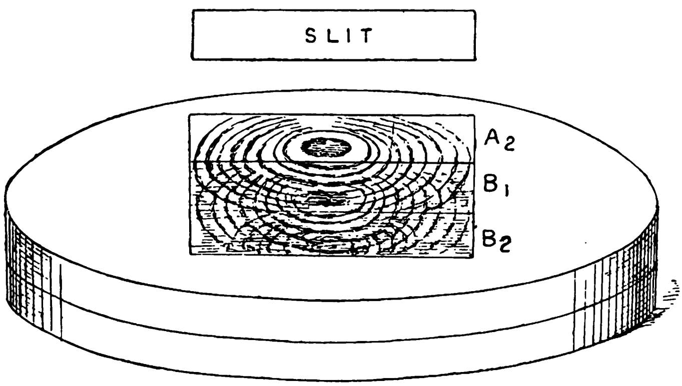


primary rings, produced by light which has escaped interference in *c* and *h* (fig. 8), but has interfered for the first

time at *s*, and has therefore the same effect as if it had been originally incident at *f*. (2) *Continuations* of the first reflexion of the primaries visible in B_1. These continuations are caused by light which has escaped interference as far as *o*, and is the result of the internal reflexion at *d*, i. e. the rings occupy the same position as if the light had been originally incident at *d*. (3) A new reflexion of the primary rings, first caused at *c*, and reflected at *d*, *g*, *f*, *s*.

Observation shows these three sets of rings in B_2 (*vide* figures 6 and 9, fig. 9 being added to make plain the arrangement by which fig. 6 was obtained), and no more. In the same way we can predict exactly what the interference systems will be like in C_1, C_2, &c. We have not yet made any attempt to examine the effect of the reflexions classed under (*d*), p. 161. If we look at fig. 8, which is part of fig. 7 on a larger scale, we see that reflexions take place at *c*, *y*, *z*, *w*, *v*: now if the path *a b c* causes the part of the ray *m a* which takes it to be an odd number of half wave-lengths behind that part of *n c* which is reflected at *c* in the direction *c d*, and thus causes interference in *c d*; then

Fig. 9.



c y z will cause the ray *a b c y z* which is reflected at *c* to be very nearly an even number of half wave-lengths behind the incident ray *o z*, and therefore this ray will increase the brightness of *z p*: similarly in *v q* the residual light which reaches *v* by the path *a b c y z w v* must be an odd number of half wave-

lengths behind the light in *s v*, and will therefore interfere: but since in every case the diminution in intensity caused by these reflexions at *a b c y z* must be rapid, it is practically only the reflexions *a b c y z* that can result in visible phenomena. If the region from *a* to *v* produces, on the whole, one of the dark rings (using monochromatic light), then if *z p*, regarded as part of *n c y z p* and of *o z*, be the darkest part of the ring, where interference is a maximum, nevertheless at *z* the part of the ray from *a*, namely *a b c y z*, will reinforce the reflected part of the ray *o z p*, and make the interference less complete than it would otherwise have been; and, similarly, a bright ring will not be quite so bright as it would be if there were no reflexions in the air-space. In short, the last-considered reflexions superimpose upon the primaries another set not in general coincident with them, and therefore in general weakening their intensity.

It is also evident, since the more refrangible the light employed, the smaller the diameters of corresponding rings, if we use white light for the generation of the rings and the subordinate systems, we can predict the result: for colour will be visible in any of the images of the slit B_1, B_2, C_1, C_2, &c. at the points of intersection of the dark curves when it has faded elsewhere. Hence co-major-axial hyperbolic broken lines of colour must result, since the various sets of curves in monochromatic light are intersecting systems of circles whose centres lie in one straight line. As the systems considered lie more and more remote from the primaries, the first hyperbola formed will be intersected by others, their numbers ever increasing, till, finally, the whole space considered is so full of them that it seems everywhere pervaded by faint white light.

The results obtained by the experiments described in this paper may be briefly stated as follows:—

(1) The method gives a very simple method of viewing Newton's rings by the light emitted from the two interior surfaces of the glass plates, free from all other light, except only that due to reflexions *in the air-space.*

(2) It reveals to the eye (for the first time) the subordinate interference systems which coexist with the primary

rings, and demonstrates which of these reflexions must be taken into account in framing the theory of the rings as they are generally viewed.

(3) It supplies a method of analysing these systems experimentally.

(4) It shows that most probably the interference of monochromatic light in forming the rings is never absolutely complete, though very nearly so.

NOTE.—The photograph (fig. 6) of the interference systems was taken by the light of sodium chloride, volatilized in a Bunsen flame. The exposures being for A_2, B_1, and B_2, 10 min., 60 min., and 240 min. respectively. The thickening of some of the lines at certain points in B_1 and B_2 is due to the faint blue light of the Bunsen burner. It is most noticeable in the immediate neighbourhood of the central spot. The plates used were Edwards's isochromatic medium, stop *f*/8. The illustration is an enlargement from $1\frac{1}{2}$ in. $\times \frac{1}{2}$ in.

Eton College, Windsor.
March 11th, 1898.

DISCUSSION.

Prof. HERSCHEL said it was rather difficult to follow the arguments of the author without witnessing the phenomena. Much complication was introduced by the successive reflexions; it was not clear what became of them. There was no doubt as to the advantage of a narrow slit for the illumination. He thought some of the secondary reflexions might be got rid of by using plates that were slightly prismatic.

Prof. THOMPSON had, in his own laboratory, verified the advantages of the author's method of illumination. The result was a very sharply-defined first system of rings. Curves of subordinate interference were easily to be observed by this arrangement.

Prof. BOYS noticed in the photograph of the ring-systems that the independent systems of bands were distorted at the points of intersection. The intersecting curves formed a sort of honeycomb, or hexagonal system, instead of a system of curvilinear quadrilaterals. This distortion reminded him of similar effects observed in the photographs of "ripples."

Mr. Edser said he had often noticed similar distortions, but he had always been able satisfactorily to explain them as being the result of imperfect focussing. The author had referred to the fact that a thin film when viewed by reflected light appears black. A phase-change of half a wave-length takes place either on reflexion at a rarer or at a denser medium; but there is no information from which to decide between these two alternatives. The truth of the assumption that the phase-change occurs at the denser medium seems to depend, so far as experimental evidence is concerned, upon the observation that in Lloyd's bands the central one is black. To produce the Lloyd's bands only one mirror is used; the bands produced by Fresnel required three mirrors. Wernicke performed an interesting series of experiments, in which white-light reflected for various angles of incidence from a thin sheet of glass was examined spectroscopically. The spectrum was crossed by numerous black bands, and from the position of these bands in the spectrum the thickness of the glass was calculated. The calculated thickness when the angle of incidence was great differed from that obtained with small angles of incidence; the conclusion was that when light is internally reflected, even at an angle of incidence less than the angle of total reflexion, a phase-change is produced. If the space between the two plates in Mr. Porter's experiment were filled with a substance of higher refractive index than glass, a confirmation, or otherwise, of this result might be obtained.

XXII. *Galvanometers.—Third Paper.*
By Prof. W. E. AYRTON, *F.R.S.*, *and* T. MATHER *.

SECTION A.

Introductory.

UNDER the above title the authors, in conjunction with Dr. Sumpner, read a paper before the Physical Society in January 1890 †, and gave a list of sensibility-records relating to a large number of instruments of various types. To facilitate comparison between different instruments, the results were reduced to a common standard as regards scale-distance, length of a scale-division, periodic time of vibration, and resistance, the values chosen for this purpose being, respectively, a scale-distance equal to 2000 scale-divisions, periodic time 10 seconds, and resistance 1 ohm. The sensibilities for steady currents were expressed as the number of scale-divisions deflexion a current of one microampere would produce (assuming proportionality) under the standard conditions, and the sensibilities for quantity as the swing produced by the instantaneous discharge of one microcoulomb when the instruments were used ballistically. The resulting numbers give a measure of the relative sensitiveness of the various instruments when quickness of action is all important. For cases where time is of little consequence, and constancy of zero (as depending on the controlling forces) is of prime importance, the results were modified by appropriately introducing the moment of inertia of the suspended systems.

Our reasons for choosing a scale-distance of 2000 scale-divisions and a period of 10 seconds as standard conditions, were, that these conditions represented more nearly than any other round numbers those under which sensitive galvanometers were generally used. For example, scales of half-

* Read May 13, 1898.

† Proceedings of the Physical Society of London, vol. x. p. 393; Phil. Mag. vol. xxx. p. 58.

millimetres used at about a metre distance were frequently met with, whilst scales of fortieths of an inch used at 3 to 4 feet distance were quite common. Further, nearly all the reflecting-instruments at the Central Technical College had long scales with divisions about $\frac{1}{24}$ of an inch placed at about 6 feet from the mirror. Since 1890, however, scales divided into single millimetres have come into much more general use, and may now be regarded as the standard type in physical laboratories. As the present tendency in galvanometer construction is to reduce the size of the needles and mirrors, and therefore the available light, it is probable that scale-distances will decrease rather than increase. We are therefore led to believe that a scale-distance equal to 1000 scale-divisions (rather than 2000) would be, on the whole, a more convenient standard to adopt. Another reason for proposing the change is to facilitate the international acceptance of a common system; for we learn from Prof. H. du Bois that our system thus modified is to be recommended for adoption at the next meeting of the Society of German Physicists and Physicians in September next at Dusseldorf. Doubtless some instrument-makers will object to the change; others, we are sure, will welcome it, for some have used the proposed standard themselves for some time. *Millimetres at a metre*, as the deflexion per microampere, is a convenient way of stating the "figure of merit" of sensitive galvanometers. Moreover, the conversion into circular measure is quite simple, 1 division at 1000 divisions distance corresponding with a movement of the suspended system through $\frac{1}{2000}$ of a radian. We would therefore suggest that in future all sensibilities be expressed in terms of this standard. In the new sensibility-records given in Table II. of the present paper we have adopted this standard; and to facilitate comparison of these records with those given in our 1890 paper, we have appropriately reduced and corrected the numbers given in the principal table of the latter paper, and embodied them in Table I.

As regards the standard periodic time, 10 seconds represented the average conditions under which sensitive galvanometers were used much more nearly than any other simple round number; hence we decided in 1890 to use this time as

the standard period to which all sensibilities were reduced. So far as we can see, this is still the most convenient standard time, and it has therefore been retained in the present paper.

The reduction to a standard resistance of one ohm has proved of considerable value in comparing various instruments, and has been frequently used by others interested in the subject. The question as to whether the "square-root law," *i. e.*

$$\text{Deflexion per microampere} \propto \sqrt{\text{resistance}},$$

or the two-fifths power law,

$$\text{Deflexion per microampere} \propto (\text{resistance})^{\frac{2}{5}},$$

should be used in this reduction is still perhaps debatable. On the whole, we consider the two-fifths law more nearly true over wide ranges of resistance; nevertheless we have given the principal results reduced according to both systems.

Recent Improvements.

Since the above-mentioned paper was published considerable advance has been made in the construction of sensitive galvanometers, more especially in the direction of reducing the dimensions of the suspended parts. We therefore thought it desirable to prepare a supplementary list of records, giving the results obtained on more recent instruments. In passing, we may remark that German and American instruments of the suspended-magnet type show the greatest progress in the reduction of dimensions and consequent quickness of action, whilst the d'Arsonval, or moving-coil type, seems to have received its greatest development in England.

To make the present lists more complete than the original one, we have inserted three additional columns, giving respectively the period of vibration of the suspended system when tested, the logarithmic decrement, and the actual sensitiveness of the instruments obtained in the test, reduced to standard scale-distance and scale-divisions. From the

first two of these columns (Columns 1 and 2 of Table II., pp. 176, 177) the relative suitabilities of the various instruments for ballistic or deflexional observation can be seen; whilst the third column (Column 3, Table II.) shows the actual "figures of merit" of the galvanometers (reduced to standard scale-distance) in the condition under which they were tested. This latter column is of considerable importance in the case of instruments whose controlling couple cannot be readily altered—for example, in the case of ordinary moving-coil instruments.

In addition to the three columns above referred to, we have also given in Table V. (p. 186) a list of records relating to moving-coil galvanometers used as voltmeters.

The values of $D/r^{\frac{1}{2}}$ or $D/r^{\frac{2}{5}}$ in columns 9 and 10 of Tables I. and II. express the relative sensibilities of the various instruments for constant period and constant resistance of coil, when the galvanometers are used as ammeters. The same columns give their relative merits as voltmeters, provided the resistance of the coil is equal to the resistance between the terminals, *i. e.* when the resistance of the connexions is small compared with that of the coil. This may be taken to be the case in instruments with fixed coils; but for moving-coil galvanometers it is by no means true, especially when the coils have low resistance, for in some instruments the resistance of the suspensions is greater than that of the coil. Further, as instruments of the d'Arsonval type cannot always have their period, and therefore their sensitiveness, easily varied, no reduction to constant period or constant resistance has been made in Table V., and the numbers in column 3 (F′) Table V. give the actual sensibilities of the respective instruments as voltmeters when the scale-distances equal 1000 scale-divisions.

We may again point out (as in our original paper *) that the comparison of galvanometers by reduction to constant period, scale-distance, and resistance, is a purely electromagnetic comparison, and takes no account of optical magni-

* Phil. Mag. vol. xxx. pp. 83-84; and Proc. Phys. Soc. vol. x. p. 420.

fication, which in some instruments is of a much more perfect nature than in others. It is also to be noticed that Table II. contains some records of galvanometers having very short periods, *e. g.* instruments numbered 4, 25, 41, 42, 43, 46, and 47; and the results of the tests on these instruments have, for the sake of uniformity, been reduced to the same period, scale-distance, and resistance, as the galvanometers having longer periods. It must not, however, be supposed that these short-period instruments could easily have their periodic time lengthened to 10 seconds—indeed, in some cases this would be practically impossible. But although the values given in columns 5, 6, 7, 12, 13, 14, 17, and 18 for the above-mentioned short-period instruments cannot as yet be realized in practice, the ratios of these numbers to those in the corresponding column for any other instrument give approximately the relative sensibilities of the two instruments compared, when both have the same *short* periodic time. To make our meaning clearer we will compare instruments numbered 24 and 43 in Table II. Referring to column 13 we find the values of $D/r^{\frac{2}{5}}$ for these instruments to be 254 and 985 respectively, indicating that No. 43 has a "factor of merit" between three and four times as large as No. 24. This must *not* be taken to mean that No. 43 is *actually* more sensitive than No. 24, but that No. 43 *would be* more than three times as sensitive as No. 24 if the period of No. 24 was reduced to, say, $\frac{1}{10}$ of a second by increasing its controlling field, and the period of No. 43 increased to $\frac{1}{10}$ of a second.

TABLE I.

1890 List of Sensitive Galvanometer Records (reduced to present standard).

	Type.	Description.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.
			T.	λ.	F.	r.	M.	D.	S.*	V.	I.	DI.	$SI^{\frac{1}{2}}$	$\frac{D}{r^{\frac{1}{2}}}$	$\frac{D}{r^{\frac{2}{5}}}$	$\frac{S}{r^{\frac{2}{5}}}$	$\frac{DI}{r^{\frac{2}{5}}}$	$\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}}$	$\frac{D}{r^{\frac{2}{5}}V}$	$\frac{S}{r^{\frac{2}{5}}V}$	$\frac{DI}{r^{\frac{2}{5}}V}$	$\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}V}$
1.	Thomson. One pair of coils.	Elliott's Tripod (F.T.C.) ‡	3·62	0·19	0·50	0·6	3·8	3·8	2·38	...	...	...	...	5	4·7	2·95						
2.		Tripod (F.T.C.) ‡	2·55	0·33	6·65	50	102	102	64	...	0·04	4·08	12·8	14·4	21·4	13·5	0·85	2·7				
3.		Elliott's Tripod, specially light needles †	20	...	30,800	3,860	7,550	7,700	4,850	125	0·01	77	485	124	283	178	2·83	17·8	2·27	1·43	0·023	0·15
4.		„ „ „ „ „ †	20	...	64,000	6,000	15,750	16,000	10,000	125	0·01	160	1,000	206·5	493	308	4·93	30·8	3·95	2·47	0·040	0·25
5.	Thomson. Two pairs of coils.	Elliott's Glass Cylinder (F.T.C.) ‡	2·33	0·26	66	4,600	1.220	1,220	760	...	...	...	...	18	41·5	26						
6.		Latimer Clark & Muirhead (F.T.C.) ‡	1·33	0·20	95	20,000	8,700	5,350	3,360	...	...	...	...	38	101·5	64						
7.		Elliott's Stock Pattern †	20	...	14,200	7,000	4,300	3,550	2,225	30	0·013	46	254	42·5	102·5	64·5	1·34	7·35	3·4	2·15	0·045	0·025
8.		„ Large Coil Pattern †	20	...	30,400	12,300	7,680	7,600	4,775	240	0·013	99	545	68·5	175	110	2·27	12·6	0·73	0·46	0·010	0·05
9.		„ „ „ „	20	...	43,600	30,000	10,710	10,900	6,850	240	0·013	142	780	63	176	110	2·3	12·7	0·74	0·46	0·010	0·05
10.		„ Special Form †	20	...	80,000	30,000	20,000	20,000	12,500	250	...	...	...	115·5	324	202·5	...	...	1·34	0·30		
11.		„ Ayrton & Perry Ballistic (Japan)	39·5	...	11,200	20,000	1,070	715	450	...	...	...	...	5·25	13·65	8·55						
12.		Langley's Bolometer	40	...	2,000	20	125	125	78	...	...	...	...	28	38	23·5						
13.	Mudford's Thomson. One pair of coils.	Simple Reflecting, Fig. 5 (C.T.C.)	5·8	...	59	26	347	175	110	72	0·042	7	22·5	33·3	47·2	30	1·96	6·1	0·66	0·42	0·028	0·08
14.		„ „ „ „	5·7	...	350	2,457	1,375	1,080	680	72	0·042	45·5	142	21·8	47·8	30	2·00	6·15	0·67	0·42	0·028	0·08
15.		„ „ „ „	5·7	...	547	6,410	2,720	1,680	1,055	72	0·042	71	218	21	50·5	31·5	2·13	6·5	0·70	0·44	0·030	0·065
16.		„ „ „ Nalder Bros.‡	20	...	3,440	2,086	880	860	540	72	...	...	...	18·8	40·5	20·5	...	...	0·56	0·36		
17.		Bell Magnet Ballistic, Nalder Bros.†	20	...	2,700	7,800	720	675	425	...	...	...	...	7·65	18·7	11·8						
18.		„ „ „ „ „ †	20	...	2,400	10,000	640	600	378	...	...	...	...	6·0	15·1	9·5						
19.		Simple Reflecting, Fig. 5 (C.T.C.)	5·7	...	370	5,330	...	1,140	720	72	0·042	47·8	148	15·5	37·0	23·3	1·55	4·45	0·51	0·33		

20.	Mudford's Thomson. Two pairs of coils.	Mudford, Fig. 6 (F.T.C.)	4·0	...	238	700	1,720	1,485	930	...	...	...	...	56	107·5	68						
21.		„ „ (C.T.C.)	7·9	...	500	686	1,660	800	505	88	0·084	67	147	30·5	59	37	4·91	10·8	0·67	0·42	0·055	0·12
22.		„ Variable damping, Large coils (C.T.C.)	10·0	...	2,140	9,744	4,200	2,140	1,320	200	0·052	111	301	21·8	54	33·5	2·81	7·65	0·27	0·17	0·014	0·038
23.		High Insulation Galv., Fig. 7 (C.T.C.)	10·0	...	27,100	360,000	50,000	27.100	17,000	236	0·01	271	1,700	45	162	101	1·62	10·1	0·69	0·43	0·007	0·043
24.		„ „ „ Nalder Bros.†	20	...	62,800	350,000	15,700	15,700	9,850	...	...	...	...	26·6	95	59·5						
25.	Rayleigh, Gray, or Rosenthal.	Rosenthal Non-Astatic, Fig. 8 (C.T.C.)	9	0·13	10·4	44	22·9	12·7	8	0·29	1·5	19	9·8	1·91	2·8	1·75	4·2	2·15	9·65	6·05	14·5	7·4
26.		„ Astatic, Fig. 10 (C.T.C.)	10·5	...	15·3	44	25	13·9	8·7	0·29	0·7	9·7	7·25	2·1	3·05	1·9	2·14	1·6	10·5	6·55	7·4	5·5
27.		„ Non-reflecting „	8	...	2·9	5·2	...	4·53	2·85	2·0	0·21	0·96	1·31	2·0	2·35	1·47	0·49	0·68	1·18	0·74	0·25	0·34
28.		Gray's Galv. (Pr. Roy. Soc. 1884, p. 287)	22	...	155	30,000	32	32	20	51	50	1600	141·5	0·19	0·52	0·3	26	2·3	0·01	0·006	0·5	0·45
29.		Ballistic „ Figs. 11 & 12 (C.T.C.)	14·4	0·035	65	4,090	54	31·5	19·6	36	17	535	82·5	0·5	1·13	0·7	19·5	3·0	0·03	0·02	0·55	0·085
30.		Threllfall's Galv. (Phil. Mag. Dec. 1889, p. 460)	80	...	120	15,680	2·90	1·87	1·18	72	...	...	...	0·015	0·039	0·025	...	...	0·00054	0·00034		
31.	d'Arsonval.	Jolin's High-resistance (C.T.C.)	1·75	...	3·98	750	137	130	81·5	2·8	30	3900	448	4·75	9·2	5·75	276	31·5	3·3	2·05	99	11
32.		Carpentier's Ordinary	0·8	...	1·34	208	286	188	118	...	...	...	...	13	22·2	14						
33.		Jolin's Low-resistance (C.T.C.)	1·5	...	0·475	3·8	39	21	13·2	0·6	5·5	115	31	10·8	12·5	7·75	67	18·2	21	13	113	30·5
34.		Invariable Sensibility, Large type (C.T.C.)	2·6	...	6·1	21	190	90·5	57	0·96	13	1180	205	19·8	27	16·8	350	61	28	17·5	365	63
35.		Nalder's type fitted with invariable sensibility device: Pole pieces in (C.T.C.)	0·63	...	0·335	274	145	84·5	53	0·2	1·5	127	64·5	5·1	8·9	5·6	13·5	6·85	44·3	28	67·5	34·3
36.		„ out „	1·55	...	2·91	274	209	121	76	0·2	...	...	...	7·3	12·8	8·1						
37.		Carpentier's Milliammeter (C.T.C.)	...	...	...	183	...	36·5	22·9	3·2	25	910	114	2·7	4·5	2·8	113	14	1·4	0·87	35	4·35
38.	Flat coil or Multiplier type.	Single-needle Post-Office pattern (C.T.C.)	2·5	...	5·95	800	...	95	60	...	...	...	...	3·35	6·55	4·1						
39.			...	...	...	400	...	151·5	95	22	0·2	30·3	42·5	7·55	13·8	8·7	2·75	3·88	0·63	0·40	0·125	0·175
40.		Suspended needle instrument with	4·0	...	11·6	350	...	73	45·9	22	0·2	14·6	20·5	3·9	7·0	4·4	1·41	1·97	0·32	0·20	0·065	0·09
41.		Coils 6 centimetres long outside	4·0	...	10	64	...	62·5	39·4	22	0·2	12·5	17·5	7·8	11·9	7·5	2·38	3·33	0·54	0·34	0·105	0·15
42.		„ 2·7 „ broad „	16	...	19	4·3	...	7·4	4·65	22	0·2	1·48	2·05	3·55	4·13	2·6	0·83	1·14	0·19	0·12	0·035	0·05
43.		„ 2·2 „ high „	...	...	...	4·3	...	17·6	11·1	22	0·2	3·52	5·0	8·5	9·8	6·2	1·96	2·25	0·43	0·28	0·09	0·125
44.		Winding 0·6 „ thick	9·6	...	3·14	0·87	...	3·4	2·14	22	0·2	0·7	0·95	3·65	3·6	2·27	0·75	1·01	0·16	0·11	0·035	0·045
45.			...	...	...	0·87	...	8·35	5·25	22	0·2	1·65	2·35	9·0	9·0	5·55	1·75	2·5	0·41	0·25	0·075	0·115
46.		Nalder's Detector, nearly astatic †	...	...	490	2,000																
47.		„ „ partially „ †	...	...	42	2,000																

* The numbers in this column are calculated from those given in column 6 without allowing for damping.
† The constants of these instruments have been kindly furnished by the makers.
‡ These instruments were tested after being in use for some time and without their needles being remagnetized.

TABLE II.

Supplementary List of Sensitive Galvanometer Records.

Number.	Type.	Description.	1. T.	2. λ.	3. F.	4. r.	5. M.	6. D.	7. S.*	8. V.	9. I.	10. DI.	11. $SI^{\frac{1}{2}}$	12. $\frac{D}{r^{\frac{1}{2}}}$	13. $\frac{D}{r^{\frac{2}{5}}}$	14. $\frac{S}{r^{\frac{2}{5}}}$	15. $\frac{DI}{r^{\frac{2}{5}}}$	16. $\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}}$	17. $\frac{D}{r^{\frac{2}{5}}V}$	18. $\frac{S}{r^{\frac{2}{5}}V}$	19. $\frac{DI}{r^{\frac{2}{5}}V}$	20. $\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}V}$
1.	Kelvin. One pair of coils.	Elliott's Tripod pattern † 1892	20	...	53	0·5	13·1	13·2	8·2	...	...	...	...	18·6	17·4	10·9						
2.		,, ,, ,, † ...... ,,	20	...	1,000	14·8	250	250	157	...	...	...	...	65	85	53·5						
3.		,, ,, ,, † ...... ,,	20	...	500	19·9	125	125	79	...	...	...	...	28	37·7	23·8						
4.		McKittrick's Oscillograph Galvanometer. (Soft-iron needle)...1896	0·000377	small	0·0014	270	296,000	985,000	620,000	16	1×10^{-6}	0·99	620	60,000	105,000	67,000	0·11	67	6,550	4,200	0·0066	4·2
5.	Kelvin. Two pairs of coils.	Elliott's glass cylinder pattern †. [1892	20	...	3,990	80·5	997	998	625	...	...	...	...	111	173	108						
6.		Queen & Co., Philadelphia. (Re-magnetized by Wadsworth). 1894	20	...	10,000	309	2,500	2,500	1,570	...	...	...	...	143	253	158						
7.		Prof. F. W. Burstall's Galv. ...1895	3	...	11	15	366	122	76	118	...	...	...	31·5	41·3	26	...	...	0·35	0·22		
8.		Hartmann & Braun, Frankfürt †. [1896	9	...	500	3,500	620	620	390	...	...	...	...	10·5	23·5	14·9						
9.	Mudford-Kelvin. Two pairs of coils.	Mudford Galvanometer. (Designed by Mather; Light needle)...1891	10·9	1·0	11,200	3,887	10,700	10,350	6,500	25	0·01	104	650	166	380	240	3·8	24·0	15·2	9·6	0·15	0·96
10.		W. B. Snow's Bolometer Galvanometer. (Small light needle).1892	20	...	22,200	140	16,500	5,550	3,500	155	...	...	...	470	770	485	...	...	5·0	3·2		
11.		E. L. & E. F. Nichols' Galvanometer. (Very small light needle) ...1892	10	...	6,650	9·3	9,800	6,650	4,180	...	...	...	...	2,200	2,730	1,700						
12.		E. L. Nichol's & W. S. Franklin's Galv. (Small light needle). ...1892	14	...	1,390	15	840	705	443	...	...	...	...	182	239	149						
13.		Ditto (ditto) .... ,,	7	...	1,390	150	3,400	2,835	1,780	...	...	...	...	231	381	240						
14.		Nalder Bros. Galv. modified by Bourne. (Vertical needle). ...1893	17	...	7,800	19,640	9,700	2,700	1,700	...	0·05	135	380	19·3	52	32·5	2·6	7·25				
15.		Elliott's Galv. (Des. by Wadsworth; very small light needle).......1894	20	...	25,000	86	6,250	6,250	3,930	600	...	...	...	675	1,050	660	..	...	1·75	1·1		
16.		Nalder Bros. Galvanometer. (Light needle) † 1895	20	...	35,500	6,490	8,900	8,900	5,550	115	...	...	...	110	265	165	...	...	2·3	1·43		
17.		Weiss' Galvanometer. (Small vertical needle) 1895	10	...	...	146	...	665	417	...	...	...	...	55	90·5	56·5						
18.		Ditto. (Very small vertical needle) .... ,,	10	...	...	...	...	...	...	...	...	...	...	600								
19.		Ditto (ditto) .... ,,	10	...	...	...	...	...	...	...	...	...	...	750								

20.	Paschen-Kelvin. Two pairs of oval coils.	Paschen's Galvanometer. (Very small light needle) .... ? 1892	30	...	23,150	20	7,000	2,580	1,620	...	...	...	...	575	775	488						
21.		Ditto (ditto) 1893	15	...	101,000	60	135,000	45,000	28,300	...	...	...	...	5,800	8,750	5,500						
22.		Above instrument after 3 years' use. (Connected up differently)...1898	17	...	48,000	15	41,700	16,700	10,500	...	...	...	...	4,310	5,650	3,550						
23.		Paschen's Galvanometer. (Very small light needle) 1896	10	...	27,000	40	27,000	27,000	17,000	...	...	...	...	4,280	6,150	3,850						
24.		Ditto, modified by Harker, Owens College. (Vertical needle) 1896	20	...	4,410	40	2,770	1,110	695	33	...	...	...	176	254	159	...	...	7·7	4·83		
25.	One oval coil.	Hotchkiss & Millis' Oscillograph Galvanometer. (Soft-iron needle)..........................1895	0·000256	small	0·000031	4·1	29,200	46,300	29,000	3·6	$2·2\times10^{-6}$	0·10	43	22,800	26,300	16,500	0·058	24·5	7,300	4,600	0·016	6·8
26.	d'Arsonval or Moving Coil.	Holden d'Arsonval. (Rectangular coil) 1890	0·8	...	0·54	26·7	129	83·5	52·5	...	...	...	..	16·1	22·4	14·1						
27.		Ayrton-Mather Ballistic. (Narrow coil) C.T.C.......................1892	2·2	0·05	32·5	13·2	1,100	665	418	0·7	0·07	46·5	110	183	238	149	16·5	39	340	213	23·5	55
28.		Ayrton-Mather, well damped. (Paul's No. 6)...................1892	4·0	2·6	47·5	300	500	500	314	...	...	...	...	29	51	32						
29.		Holden d'Arsonval. (Rectangular coil) 1893	1·42	1·77	0·39	2·22	26·4	25·4	15·0	...	...	...	...	17·2	18·1	11·3						
30.		Queen & Co., Philadelphia. (Large narrow-coil) 1893	aperiodic	...	200	178																
31.		Sullivan's "Universal" Galvanometer §..........................1895	4·61	...	288	1,260	5,100	1,325	830	...	...	...	...	37·3	76	47·5						
32.		Hartmann & Braun, Frankfurt† 1896	aperiodic	...	285	1,600																
33.		Crompton's Large pattern. (Circular coil) 1896	12·1	small	262	103·3	357	179	112	...	...	...	...	17·6	27·8	17·5						
34.		Ditto (do.) (Camb. Eng. Lab.) „	16·1	0·03	138	100	55	53	33·3	3·12	15·6	825	131	5·3	8·4	5·25	131	20·8	2·7	1·68	42	6·65
35.		Ditto (do.) (W. R. Cooper, Esq.)1897	7·2	0·018	70	62·6	135	135	85	...	...	...	...	17·1	25·8	16·2						
36.		Ditto Midget pattern (do.) (Camb. Eng. Lab.) 1897	10·0	small	66·3	57·6	77	66·3	41·7	1·33	5·9	391	101	8·75	13·1	8·2	77	19·5	9·8	6·15	58	14·7
37.		Nalder Ballistic. (Rectangular coil). (Camb. Eng. Lab.) 1897	12·9	0·01	760	1,053	428	455	286	...	...	...	...	14·0	28·5	18						
38.		Nalder Damped (do.) No. 6283.1897	3·25	0·264	3·62	29·5	3·12	34·3	21·5	...	...	...	...	6·3	8·85	5·55						
39.		Andersen's Galvanometer. (Circular coil) 1897	4·7	2·63	6·45	257	97·6	64	40·2	...	...	...	...	4·0	6·05	4·38						
40.		Ditto (do.) .... „	4·5	1·62	3·88	32·4	48·5	31·8	20	...	...	...	...	5·55	7·9	5·0						
41.		Ayrton-Mather Speaking Galv. (A) (Narrow coil) C.T.C.1897	0·0965	...	0·55	214	72 0	5,850	3,625	0·54	0·034	190	675	375	650	408	22·1	75	1,200	755	41	139
42.		Ditto (B) „ „ „	0·0868	...	0·52	267	8800	6,900	4,325	0·54	0·036	248	820	421	735	463	26·5	87·5	1,365	855	49	163
43.		Ditto (C) „ „ „	0·0925	...	0·76	213	11200	8,900	5,600	0·54	0·031	276	980	570	985	623	31·4	109	1,830	1,150	53·5	207
44.		Ayrton-Mather New pattern Galv. (Narrow coil) C.T.C. 1897	5·84	0·07	230	22·2	965	680	424	...	...	...	...	145	197	124						
45.		Ditto „ „	7·6	...	102	1·9	250	177	111	0·6	0·30	53	56	128	137	86	41	43·3	228	143	68	72
46.		Duddell's Oscillograph. (Very narrow coil and electromagnet). 1897	0·000358	...	0·000156	1·3	164,500	122,000	76,500	...	about 30×10^{-6}	...	...	107,000	109,000	68,500	3·3	374				
47.		Ditto „ „ 1898	0·000093	...	0·00042	2·08	1,920,000	4,780,000	3,000,000	...	6×10^{-6}	...	...	3,300,000	3,560,000	2,240,000	0·21	548				

* The numbers in this column are calculated from those given in column 6 without allowing for damping.

† The constants of these instruments have been kindly furnished by the makers.

§ The inventor of this galvanometer says its period was intentionally made long for a special purpose and that the period could be considerably reduced without lessening the deflection per microampere.

Explanation of Tables I. and II.

Column 1 (or T) gives periodic time of vibration in seconds when tested.
„ 2 (or λ) „ logarithmic decrement of motion when tested.
„ 3 (or F) „ deflexion in divisions per microampere (as tested) when scale-distance = 1000 divisions.
„ 4 (or r) „ resistance of coil in ohms.
„ 5 (or M) „ deflexion in millimetres per microampere when undamped period is 10 seconds and scale placed as in actual use of the instrument.
„ 6 (or D) „ deflexion in divisions per microampere when undamped period = 10 seconds and scale-distance = 1000 divisions.
„ 7 (or S) „ swing per microcoulomb under same conditions as in last.
„ 8 (or V) „ volume occupied by convolutions of wire, in cubic centimetres (approximately).
„ 9 (or I) „ moment of inertia of the suspended system, in C.G.S. units (approximately).
„ 10 (or DI) „ deflexion in divisions (scale-distance = 1000 divisions) per microampere for constant controlling moments, and for a periodic time equal to $10\sqrt{I}$ seconds.
„ 11 (or $SI^{\frac{1}{2}}$) „ swing per microcoulomb under same conditions as in last.

Columns 12, 13, and 14 give the deflexion per microampere and swing per microcoulomb, when the period is 10 seconds and resistance of each instrument is one ohm*.

Columns 15 and 16 give the deflexion per microampere and the swing per microcoulomb, for the same controlling moment, the resistance of each instrument being one ohm.
„ 17 „ 18 give the deflexion per microampere and the swing per microcoulomb per cubic centimetre of coil, when the period is 10 seconds and the resistance of each galvanometer one ohm.
„ 19 „ 20 give the deflexion per microampere and the swing per microcoulomb per cubic centimetre of coil, for the same controlling moment, and the resistance of each instrument equal to one ohm. Periodic time of any instrument is $10\sqrt{I}$ seconds.

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* Column 12 is based on the assumption that the sensitiveness is proportional to the square root of the resistance, whilst columns 13, 14, 15, &c. assume that sensitiveness varies as the two-fifths power of the resistance. (See Phil. Mag. July 1890, p. 85 *et seqq.*, and Proc. Phys. Soc. vol. x. p. 422.)

Comparison as Ammeters.

On comparing Table II. of the present paper with the one published in 1890 (shown in Table I., pp. 174, 175), it will be observed that in columns 12 and 13 (which express the factors of merit of the various instruments when time is of importance) the values are on the average greatly increased. For example, the highest values in the 1890 table were given by one of Messrs. Elliott Bros.' Thomson instruments of special form (see line 4, Table I.), for which $D/r^{\frac{1}{2}}$ was 206·5 and $D/r^{\frac{2}{5}}$ 493, and these numbers were far above any others in the list made out at that time. These figures are now surpassed by no less than eleven instruments given in the present list (omitting the oscillograph records in lines 4, 25, 46, and 47); and three of the eleven instruments are of the moving-coil type.

The highest values in the present table (omitting the oscillographs previously mentioned) are those given by one of Prof. Paschen's instruments, line 21, Table II., and are 5800 and 8750 for $D/r^{\frac{1}{2}}$ and $D/r^{\frac{2}{5}}$ respectively. We therefore see that Prof. Paschen has produced an instrument which, at a given periodic time and of a given resistance, is about twenty times as sensitive as the best given in our previous paper. Other instruments of the moving-magnet type giving very high factors of merit are:—

(*a*) Two Paschen instruments. (Lines 20 and 23 of table.)
(*b*) Prof. Nichols' galvanometer *. (Line 11 of table.)
(*c*) Smithsonian Institution instrument, designed by Wadsworth † and constructed by Messrs. Elliott Bros. (Line 15 of table.)
(*d*) Prof. W. B. Snow's bolometer galvanometer ‡. (Line 10 of table.)
(*e*) Two galvanometers by Weiss §. (Lines 18 and 19 of table.)

To show at a glance the character of the improvement, the

* 'The Galvanometer,' by Prof. E. L. Nichols, p. 80.
† Phil. Mag, vol. xxxviii. p. 553.
‡ Physical Review, vol. i. p. 37.
§ *C. R.* vol. cxx. p. 728, and *Journ. Phys.* vol. iv. p. 212.

numbers for these instruments are abstracted from the main table and put in order of magnitude below.

TABLE III.

Description.	Values of		
	r.	$D/r^{\frac{1}{2}}$.	$D/r^{\frac{2}{5}}$.
Paschen's Galvanometer	60	5,800	8,750
,, ,, ...	40	4,280	6,150
Nichols' ,, ...	9·3	2,200	2,730
Weiss' ,, ...	......	750	
Wadsworth's ,, ...	86	675	1,050
Weiss' ,, ...	......	600	
Snow's ,, ...	140	470	770
Elliott's Galvanometer (from Table I.)	6,000	206·5	493

For the moving-coil instruments, the numbers are

Description.	r.	$D/r^{\frac{1}{2}}$.	$D/r^{\frac{2}{5}}$.
Ayrton-Mather Galvanometer	243	570	985
,, ,, ,, ...	267	421	735
,, ,, ,, ...	244	375	650
Invariable Sensibility Galvanometer (from 1890 Table)	21	19·8	27

These increases in the factors of merit have been brought about chiefly by reducing the dimensions of the suspended parts, and the use of better magnet steel has probably contributed something to the improvement.

The three moving-coil instruments which have surpassed the highest values of $D/r^{\frac{1}{3}}$ and $D/r^{\frac{2}{5}}$ given in our 1890 table are of the narrow coil type, the cross-section of the winding being

of the form described before this Society in 1890*, viz., two equal circles in contact.

To attain the results here shown (Table III.), the transverse dimensions of the coils have been made smaller than usual and very powerful permanent magnets employed. When we compare the records of these instruments with the best listed in our 1890 table (particulars of which are given in the last line of the preceding table) the improvement is evident; for whereas the highest value of $D/r^{\frac{2}{5}}$ given by a moving-coil instrument in 1890 was 27, the present highest value is 985.

It is also interesting to notice that so far as absolute sensitiveness for current is concerned (*i.e.* the "millimetres at a metre" per microampere, unreduced for period and resistance) the progress made with moving-coil instruments during the last eight years has been considerable. In our 1890 Table the d'Arsonval of greatest absolute sensitiveness as an ammeter was the "Large invariable sensibility" instrument given in line 4 of the d'Arsonval section of that list (see line 34, Table I.). This galvanometer gave 6·1 divisions per microampere, its period being 2·6 seconds, and coil-resistance 21 ohms. In the more recent list, Table II., the above record is surpassed by no less than thirteen instruments out of the twenty-one tabulated, the most sensitive d'Arsonval galvanometer to current being one made by Messrs. Nalder Bros. & Co., which gives 760 divisions per microampere, its period being 12·9 seconds and resistance 1053 ohms (see line 37, Table II.).

On comparing column 15 of Table II. with the corresponding one in the 1890 Table, which gives the values of $DI/r^{\frac{2}{5}}$, *i. e.* numbers proportional to the deflexion per microampere with *constant controlling moment* †, we observe that no increase is perceptible. On the contrary, the numbers in the columns of the present table are on the average smaller than in the previous one. This is what might be expected from a diminution of the dimensions of the moving parts of the

* "On the shape of Movable Coils," &c., Proc. Phys. Soc. vol. x. p. 376.

† See paper on "Galvanometers," Phil. Mag. July 1890, and Proc. Phys. Soc. vol. x. p. 421.

instruments. An examination of columns 16 in the two tables (values of $SI^{\frac{1}{2}}/r^{\frac{2}{3}}$) leads to a conclusion similar to that deduced from a comparison of columns 15, the only instance of increased values being in the three narrow-coil d'Arsonval instruments just referred to.

It is perhaps worth while again calling attention to the fact of the superior sensitiveness (as ballistic instruments) of galvanometers with suspended parts having small inertia, a fact which is clearly shown by a study of columns 9 and 14 of the tables.

Prof. R. Threlfall's "New" Galvanometer.

Since the paper was read our attention has been directed to a very sensitive galvanometer designed by Prof. R. Threlfall, M.A. (an account of which is given in the Phil. Trans. of the Roy. Soc. 1896, vol. clxxxvii. pp. 80–97), and used by him and Mr. J. H. D. Brearley in their "Researches on the Electrical Properties of Pure Sulphur." This instrument is in many respects a remarkable one. Unfortunately the details of the sensibility tests given in the printed paper are hardly sufficient to enable us to make the reductions to constant period and scale distance, as is done in Table II., so we give the results separately below. The tests were made by observing the throw produced on reversing a small current through the galvanometer, the throws being read by a telescope at about 267 centimetres from the galvanometer. The telescope had a micrometer eyepiece and a glass scale in it divided to fifths of a millimetre.

Date.	Period in seconds.	Current in amperes.	Throw on reversal in micrometer divisions.	Amperes per throw of 1 division.
Oct. 17th, 1892.	14·5	$2{\cdot}74 \times 10^{-11}$	7·9	$3{\cdot}5 \times 10^{-12}$*
Sept. 1893	25	$2{\cdot}74 \times 10^{-12}$	19·3	$1{\cdot}43 \times 10^{-13}$†

* Phil. Trans. vol. 187, line 1 of table vi. p. 95.
† Ibid. line 1 of table vi. p. 96.

The definition of the optical system was sufficiently good to permit the micrometer-scale being read to a fifth of a division, so that currents of one-fifth the magnitude of those given in the last column of the above Table could be detected.

Oscillographs.

Oscillographs, or instruments for showing the character of rapidly-varying currents or potential-differences, were practically unknown when our 1890 paper was read. They have become possible by the great reduction of dimensions of moving parts which has gone on continuously since that date. An oscillograph may be defined as a galvanometer of very short period. The short periodic time is usually obtained by using very small moving parts and a very strong control. In these instruments the reduction of size has been carried far beyond what has yet been attempted for ordinary scale-reading work. Mr. McKittrick's * galvanometer has a moving system whose moment of inertia is about one-millionth of a C.G.S. unit, or about one ten-thousandth the inertia of an ordinary *light* astatic needle fitted with a $\frac{3}{8}$ in. *light* mirror. The record of this instrument is shown in line 4, Table II., and line 25 gives the record of another oscillograph galvanometer by Messrs. Hotchkiss and Millis †. Both the above instruments are of the moving magnet type with soft-iron needles adopted for similar purposes by Blondel, Moler, and others. Two other oscillographs are given in lines 46 and 47 of Table II., which are of the moving-coil type, and were made by Mr. W. Duddell ‡ of the Central Technical College.

For convenience of reference the principal numbers relating to oscillographs are tabulated separately below, reductions being made to a constant period of a thousandth of a second, instead of 10 seconds, and the deflexions stated in terms of an ampere, instead of a microampere.

* Trans. A. I. E. E. vol. xiii. Nos. 6 and 7.

† Phys. Review, vol. iii. p. 49. In Phys. Rev. vol. iv. p. 128, Mr. Millis gives an account of another instrument of very short period. Unfortunately the data are insufficient to permit of comparison with other oscillographs.

‡ See "Oscillographs," Electrician, Sept. 10, 1897, p. 638.

TABLE IV.

Description.	t.	f.	r.	I approximately.	$d/r^{\frac{1}{2}}$.	$d/r^{\frac{2}{5}}$.
Hotchkiss and Millis' galv., 1895	0·256	31	4·1	$2{\cdot}2 \times 10^{-6}$	228	263
McKittrick's galv., 1896	0·377	1400	270	$1{\cdot}0 \times 10^{-6}$	600	1050
Duddell's galv., 1897 .	0·358	156	1·3	30×10^{-6}	1070	1090
,, ,, 1898 .	0·093	420	2·08	6×10^{-8}	33,000	35,600

Explanation of Table IV.

Column t gives the periodic time in thousandths of a second when tested.
,, f ,, deflexion in divisions per ampere as tested when scale-distance = 1000 scale-divisions.
,, r ,, resistance of instrument in ohms.
,, I ,, moment of inertia of the moving systems in C.G.S. units approximately.
Columns $d/r^{\frac{1}{2}}$ and $d/r^{\frac{2}{5}}$ give the deflexions per ampere when the periodic time is one thousandth of a second and the resistance of each instrument is one ohm. Column $d/r^{\frac{1}{2}}$ is based on the assumption that the sensitiveness varies as the square root of the resistance, whilst column $d/r^{\frac{2}{5}}$ assumes that sensitiveness is proportional to the two-fifths power of the resistance.

On comparing Mr. Duddell's 1897 instrument with those of Messrs. Hotchkiss and Millis and Mr. McKittrick, we find its factor of merit, as indicated by $d/r^{\frac{1}{2}}$ (the expression most suited to instruments having comparatively low resistance), is decidedly greater than either of the other two, in spite of the fact of its wires being of phosphor-bronze instead of copper. It may also be pointed out that the moment of inertia of the moving parts of Mr. Duddell's 1897 oscillograph is very considerably greater than the inertias of the two first-mentioned instruments, being in the approximate ratios of 30 : 2·2 : 1, thus showing the great superiority of the moving-coil type for work of this nature, where short periodic time (or high frequency) and good sensitiveness are necessary. In his latest oscillograph Mr. Duddell has carried the reduction of dimensions much further than has previously been attempted,

and has produced an instrument whose periodic time is less than a ten-thousandth of a second (frequency 10,680). Although its resistance is little more than 2 ohms, it actually gives 420 divisions per ampere at standard scale-distance, and its factors of merit, as represented by $d/r^{\frac{1}{2}}$ and $d/r^{\frac{2}{5}}$, are more than 30 times those of any instrument made previously.

Addition made October 1898.

[We desire to mention that all the oscillographs referred to in this subsection are modifications of forms first described by Mons. A. Blondel, *Comptes Rendus*, 1892, vol. cxvi. pp. 502 & 748, where both the principles and theory of Oscillography are fully dealt with. M. Blondel has kindly furnished us with the following approximate particulars relating to one of his moving magnet instruments made for workshop use :—

Periodic time of vibration (undamped) $\frac{1}{6000}$ second.
Resistance 7 ohms.
Deflexion per ampere at 1 metre=40 millim.

$$\text{Dimensions of needle}\begin{cases}\text{Iron} & 7\times1\times0{\cdot}2 \quad \text{millim.}\\ \text{Mirror} & 7\times1\text{ to }1{\cdot}5\times0{\cdot}5 \quad ,,\end{cases}$$

It should be pointed out that this particular oscillograph was designed with a view of obtaining plenty of light, so that the curves could be shown on a screen by aid of a glow-lamp, rather than for large sensitiveness.

Expressing the results in the form adopted in Table IV. we get the following values :—

t.	f.	r.	I (approximately).	$d/r^{\frac{1}{2}}$	$d/r^{\frac{2}{5}}$
$0{\cdot}1\dot{6}$	40	7	25×10^{-6}	540	660

the meanings of t, f, r, I, and d being as mentioned in the "Explanation of Table IV."]

d'Arsonval Galvanometers as Voltmeters.

In Table V. the actual sensibilities of various d'Arsonval galvanometers when used as voltmeters are given, reductions having been made to bring them to the same scale-distance and scale-divisions only.

TABLE V.—List of d'Arsonval Galvanometer Records (as Voltmeters).

Number on list.	Description.		1	2	3	4	5
			T.	λ.	F'.	r.	r'.
26	Holden d'Arsonval (Rectangular coil)	1890	0·8	...	0·020	26·7	27·2
27	Ayrton-Mather Ballistic (Narrow coil) C.T.C.	1892	2·2	0·05	1·31	13·2	24·8
28	„ Damped (Paul's No. 6)	1892	4·0	2·6	0·158	300	304
29	Holden d'Arsonval (Rectangular coil)	1893	1·42	1·77	0·136	2·22	2·86
30	Queen & Co., Philadelphia (Large narrow coil)	1893	Aperiodic	...	1·13	...	178
31	Sullivan's "Universal" Galvanometer *	1895	4·61	...	0·224	...	1260
32	Hartmann & Braun's, Frankfürt †	1896	Aperiodic	...	0·175	1600	
33	Crompton's Large pattern (Circular coil)	1896	12·1	Small	1·86	103·3	143·3
34	„ „ „ (Camb. Eng. Lab.) (Circular coil)	1896	16·1	0·03	0·975	100	141
35	„ „ „ (W. R. Cooper, Esq.) („)	1897	7·2	0·018	0·51	62·6	68·5
36	„ Midget pattern (Camb. Eng. Lab.) („)	1897	10·0	Small	1·12	57·6	59·2
37	Nalder Ballistic (Camb. Eng. Lab.) (Rectangular coil)	1897	12·9	0·04	0·715	1053	1060
38	„ Damped No. 6283 (Rectangular coil)	1897	3·25	0·264	0·108	29·5	33·4
39	Andersen's Galvanometer (Circular coil)	1897	4·7	2·63	0·025	...	257
40	„ „ („)	1897	4·5	1·62	0·12	...	32·4
41	Ayrton-Mather Speaking Galvanometer (A) (Narrow coil) C.T.C.	1897	0·0965	...	0·0022	244	244
42	„ „ „ (B) („) „	1897	0·0868	...	0·0020	267	267
43	„ „ „ (C) („) „	1897	0·0925	...	0·0031	243	243
44	„ New pattern (Narrow coil) C.T.C.	1897	5·84	0·07	6·55	22·2	35·1
45	„ „ „ „	1897	7·6	...	17·7	1·9	5·75
46	Duddell's Oscillograph (Very narrow coil and electromagnet) ‡	1897	0·000358	...	0·00012	1·3	1·3
47	„ „ („ „ „) ‡	1898	0·000093	...	0·00020	2·08	2·08

* See Note § Table II.
† The constant of this instrument was kindly supplied by the makers.
‡ These instruments have their moving coils of phosphor bronze, whereas all the others are of copper or silver.

Explanation of Table V.

Column 1 (or T)	gives the	periodic time of vibration of coil when tested (in seconds).
„ 2 (or λ)	„	logarithmic decrement of motion when tested.
„ 3 (or F′)	„	deflexion in divisions per microvolt when scale-distance = 1000 scale-divisions.
„ 4 (or r)	„	resistance of coil of instrument in ohms at about 15° C.
„ 5 (or r')	„	resistance of instrument between terminals at about 15° C.

As will be seen from the table, considerable advance has been made in recent years. Of the d'Arsonval galvanometers mentioned in our 1890 paper, the most sensitive instrument as a voltmeter we had then seen was one of the "invariable sensibility" type, having a very large magnet. Its record is given in line 4 of the d'Arsonval section of that table (see line 34, Table I.). The instrument had a period of 2·6 seconds, coil-resistance of 21 ohms, and total resistance of 57·5 ohms, and gave a deflexion of 0·105 division per microvolt when the scale-distance was equal to 1000 scale-divisions. Referring to Table V. of the present paper, we notice that in 1892 a narrow-coil galvanometer was constructed, having a period of 2·2 seconds, coil-resistance 13·2 ohms, and total resistance 24·8 ohms, which gave 1·31 divisions per microvolt, an increase of over 12 times in sensitiveness, with a shorter period.

In 1893 Messrs. Queen and Co., of Philadelphia, produced a galvanometer of 178 ohms resistance, which gave 1·13 divisions per microvolt, whilst in 1896 Messrs. Crompton showed to the authors one of their high-grade instruments, having a total resistance of 143 ohms, which gave 1·83 divisions per microvolt, its period being 12·1 seconds (see line numbered 33, Table V.). Up to the end of 1896 the Crompton instrument was the most sensitive d'Arsonval voltmeter we had any record of, but its period was, for some purposes, inconveniently long.

Table VI. is an abstract of Table V. arranged chronologically, showing improvements since 1888.

TABLE VI.

Description.	Date.	Period in seconds.	Divisions per microvolt.	Resistance of coil in ohms.	Total resistance in ohms.
1. Invariable Sensibility Type (Large)	1888	2·6	0·105	21	57·5
2. Ayrton-Mather Narrow coil	1892	2·2	1·31	13·2	24·8
3. Queen and Co., Philadelphia	1893	Aperiodic.	1·13	......	178
4. Crompton High Grade.	1896	12·1	1·83	103·3	143·3
5. Ayrton-Mather Narrow coil	1897	5·84	6·55	22·2	35·1
6. Ayrton-Mather Narrow coil	1897	7·6	17·7	1·9	5 75

During 1897 several fairly low-resistance d'Arsonval instruments of the narrow-coil type were made at the Central Technical College for use with the Lorenz apparatus tested by Prof. J. Viriamu Jones and one of the authors for the McGill University, Montreal. Records of two of them are given in lines numbered 44 and 45 of Table V. and lines 5 and 6 of Table VI. From these it will be seen that one gave 6·55 divisions per microvolt, and the other 17·7 divisions per microvolt, the latter number being nearly ten times as great as the previous best. Further particulars of these two very sensitive instruments when used as ammeters are given in lines 44 and 45 of Table II.

Uniformity in recording Tests.

We would here mention that in some of the galvanometer records we have received, the time of a single swing from one side of the scale to the other has been given as "the period," and this has caused considerable trouble on account of the sensibilities under standard conditions, calculated out on the assumption that the "complete period" was meant, coming out abnormally high. We therefore think it important that in all records of tests the "complete period," *i. e.* the time between two transits of the spot across the zero *in the same direction,* should be given. We may also remark that in all cases the logarithmic decrement (or the decrement) under the conditions in which the period was tested, should be

stated, so that the change of period caused by damping may be allowed for. Unless this be done, well-damped instruments, which are usually the most convenient, will be unfairly handicapped. In the case of d'Arsonval or other instruments, in which the periodic time cannot readily be altered by the user, it is important that the actual period of the vibration when tested should be given.

For a complete record the following particulars are required :—

(1) Resistance of instrument*.
(2) Periodic time when tested.
(3) Divisions deflexion per microampere (or microamperes per division).
(4) Scale-distance.
(5) Length of one scale-division.
(6) Logarithmic decrement under test conditions.
(7) Moment of inertia of the suspended system.
(8) Mass of suspended system.
(9) Diameter of mirror.
(10) Length of suspended magnets (perpendicular to axis of rotation).
(11) Type of instrument.
(12) Volume of coils.
(13) Date of test.

Nomenclature relating to Dead-beat Galvanometers.

Some ambiguity at present exists in the use of the words "dead-beat" as applied to galvanometers. Sometimes it is used as synonymous with "well-damped" or "aperiodic," whilst at other times the adjective is applied to instruments having quick-moving systems irrespective of whether the decrement of the motion is large or small. Since quickening the movement by increasing the control (*i. e.* decreasing the periodic time of vibration) of a given system lessens the decrement, the two uses of the words "dead-beat" are to some extent incompatible. Maxwell used the term for galvanometers in which the motion of the suspended system was "aperiodic,"

* If instrument be of the moving-coil type the resistances of coil and suspensions should be stated separately.

i. e. the system only passed once through the position of equilibrium before coming to rest. As this meaning of the word seems most rational, we think it desirable it should be retained, and that its use in connexion with instruments having rapidly moving systems whose motion is not aperiodic should be discontinued. The expressions "quick-moving" or "short-period" galvanometer might be used to denote this class of instrument.

Insulation of Coils and Terminals of Galvanometers, and the use of Price's "Guard-Wire."

When testing very high resistances, such as the insulation resistance of a short length of good cable, with a galvanometer, the difficulties which arise from leakage over the cable ends or from the galvanometer itself are well known ; the importance of constructing galvanometers so as to ensure good insulation was referred to in our previous paper *.

Since that date Mr. W. A. Price † has suggested the use of a "guard-wire" which practically eliminates error due to surface leakage over cable ends. The method there described can be adapted to galvanometers, thereby making the question of *perfect* insulation of coils and terminals of far less consequence than formerly.

Mr. Price's method consists in winding a bare wire W about midway between the extremities of the pared ends of the cable as shown in fig. 1, and connecting the other end of the wire to H, the battery side of the galvanometer G.

As the current through the galvanometer is very small, the P.D. between the wire W and the conductor C is also very small, and the surface leakage between them is quite negligible if the surface of the dielectric between them is fairly clean and dry. The full P.D. exists between W and the sheathing or wet braiding S, but any leakage-current from one to the other does not pass through the galvanometer. In the same

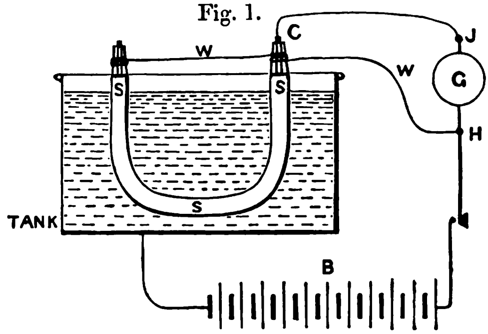
* See p. 71, Phil. Mag. July 1890.

† 'Electrical Review,' vol. xxxvii. p. 702 (1895). See also Appleyard on "Dielectrics," Proc. Phys. Soc. vol. xiv. pp. 257, 264, where possible error is evaluated.

manner error due to leakage from the terminal J of the galvanometer may be eliminated.

It will be evident that there is no need to have the battery side of the galvanometer *perfectly* insulated: all that is necessary is that the insulation resistance of H from earth is large compared with the internal resistance of the battery.

Fig. 1.



Probably the simplest way of applying the "guard-wire" principle to a galvanometer is to put one terminal of the winding to frame and case (preferably of metal), and carry the other terminal from the frame on a support moderately well insulated therefrom. The whole galvanometer may be sufficiently well insulated from earth by putting short ebonite sleeves, or caps, on the tips of the levelling screws. This construction was carried out in the Ayrton-Mather galvanometer as made by Mr. R. W. Paul in 1892, an instrument of this class being shown before this Society on June 10th in that year *. By using the frame terminal as the battery side of the galvanometer, the frame acts as an efficient guard-wire. Another advantage of the construction here described is that electrostatic deflexion of the moving system is at the same time prevented, a matter of considerable importance when resistance tests with fairly large potential-differences are being made.

A similar construction is, of course, applicable to shunt-boxes.

In concluding this section of our paper, we desire to thank

* 'Electrician,' vol. xxix. June 17th, 1892, p. 174.

the various manufacturers and inventors who have kindly supplied us with information and data relating to their instruments, and also to acknowledge our great indebtedness to C. G. Lamb, Esq., M.A., of the Cambridge Engineering Laboratory, and W. R. Cooper, Esq., M.A., for supplying us with several important records by which we have been able to add materially to the completeness of our lists.

Section B.

Limiting Sensitiveness of Thomson Galvanometers.

At the Oxford meeting of the British Association (1894), Prof. Schuster read a paper "On the Construction of Delicate Galvanometers," in which inquiry is made as to the smallest current that can be detected by a non-astatic instrument of 1 ohm resistance, and controlled by a field of strength 0·17 C.G.S. unit. The calculated minimum current depends on the resolving power of the mirror employed, the smallest angle observable being taken as $\frac{\lambda}{2d}$ where λ is the wave-length of light and d the diameter of the mirror. The result arrived at is that $1{\cdot}5 \times 10^{-8}$ amperes is the smallest current that can be detected under the assumed conditions.

Taking λ as $5{\cdot}9 \times 10^{-5}$ centimetres we find the minimum angle to be $2{\cdot}95 \times 10^{-5}$ radians or about 6·1 seconds of arc; this corresponds with 0·0593 division on a scale at 1000 divisions distance. Expressing Prof. Schuster's result in this way we get 3·9 divisions per microampere as the limiting sensibility of a non-astatic galvanometer of 1 ohm resistance when controlled by a field of 0·17 C.G.S. unit, the cavity within the coil being a sphere of 1 centimetre diameter. It is interesting to notice that to obtain a deflexion of 200 divisions per microampere (a number one might fairly expect to get from a good Thomson instrument having a period of 10 seconds and needles about a centimetre long) would necessitate the controlling field of a non-astatic instrument being reduced to less than one-fiftieth of the assumed value, or to 0·003 C.G.S. unit.

In his calculations Prof. Schuster had no need to take into account the period of vibration of the suspended system, but

in the following treatment of the subject period is considered an important factor, and the specific magnetism of the needle is also taken into account. Further, the suspended system is supposed astatic, but this is not essential, for a non-astatic system with an adjustable control would lead to the same results.

The calculations in this section of the paper were made to ascertain whether some unusually high records of galvanometer tests which had been sent to us could possibly be correct. The answer was in the affirmative. A similar investigation had previously been made relating to some instruments of the d'Arsonval type, and this gave an answer in the negative. The calculated maximum sensibility came out about $\frac{1}{20}$ of the published value. Subsequent tests made on an actual instrument gave a result about $\frac{1}{30}$ of the published value, or about two-thirds of the calculated maximum.

In treating the subject mathematically the following assumptions are made :—

(1) Clearance between coil and end of magnets 1 millimetre.
(2) Mirror and stem of negligible inertia compared with that of magnets.
(3) Perfect astaticism of needle. (Thickness negligible.)
(4) Coil unlimited in size and of best shape as voltmeter.
(5) Thickness of insulation on wire negligible.
(6) Control of suspension negligible.
(7) Deflecting field equal to field at centre of coil.

Note.—All the assumptions except (7), and perhaps (1), will give too high a value for the calculated sensitiveness.

Let $2l$ = length of magnets.
A = area of cross-section of magnets.
B = induction density in the steel.
Δ = density of steel.
m = strength of pole.
σ = specific magnetism of needles.
τ = time of vibration (complete).

Then,

$$\text{Magnetic moment} = 2ml,$$
$$= \text{M (say)},$$
$$\therefore \ \text{M} = \frac{2\text{AB}l}{4\pi}; \quad . \quad . \quad . \quad . \quad (8)$$

$$\text{Mass} = 2Al\Delta,$$

$$\text{Moment of inertia } I = \tfrac{2}{3}Al^3\Delta\,; \quad \ldots \ldots \quad (9)$$

$$\text{Specific magnetism} = \frac{B}{4\pi\Delta}\,; \quad \ldots \ldots \quad (10)$$

$$\therefore \quad \sigma = \frac{B}{98} \text{ approximately.}$$

For an astatic pair we have

$$\tau = 2\pi\sqrt{\frac{2I}{MH_1 - MH_2}}$$

where H_1 and H_2 are the controlling fields acting on the two magnets respectively;

$$\therefore \quad H_1 - H_2 = \frac{8\pi^2 I}{\tau^2 M}$$

$$= \frac{8}{3}\frac{\pi^2 l^2}{\tau^2\sigma} \quad \ldots \ldots \quad (11)$$

from (8), (9), and (10).

This gives the difference of the two fields for any periodic time of vibration, length of needle, and specific magnetism.

If θ be the deflexion produced by a current γ, then

$$\text{Controlling couple} = (H_1 - H_2)\,M \sin\theta,$$

and

$$\text{Deflecting couple} = 2M\gamma G \cos\theta,$$

where G is the principal constant of the coil.

Hence

$$\gamma = \frac{(H_1 - H_2)}{2G}\tan\theta,$$

$$= \frac{4\pi^2 l^2}{3\tau^2\sigma G}\tan\theta \quad \ldots \quad (12)$$

from (11).

Now

$$G = N\frac{1}{a^2\beta^2}\left(\frac{1}{a} - \frac{1}{x}\right) \quad \ldots \quad (13)$$

(see Maxwell, vol. ii. p. 325, 1st ed.);

where

$$N = 2\pi\int_0^\pi (\sin\theta)^{\frac{5}{2}}\,d\theta$$

$$= 9{\cdot}034.$$

α the ratio of the radius of the wire to the maximum radius of the layer;

β the ratio of the distance between the centre of two adjacent wires to the radius of the wire;

a the maximum radius of cavity in coil;

x the outer radius of coil.

The resistance of the coil is given by

$$\mathrm{R}=\mathrm{N}\frac{\rho}{\pi\alpha^4\beta^2}\left(\frac{1}{a}-\frac{1}{x}\right), \quad . \quad . \quad (14)$$

ρ being the specific resistance of metal used in the winding;

$$\therefore \quad \frac{\mathrm{G}}{\sqrt{\mathrm{R}}}=\frac{1}{\beta}\sqrt{\mathrm{N}\frac{\pi}{\rho}\left(\frac{1}{a}-\frac{1}{x}\right)} \quad . \quad . \quad (15)$$

A little consideration will show that $\frac{\mathrm{G}}{\sqrt{\mathrm{R}}}$ gives the value of G for a galvanometer whose resistance is one ohm.

This is a maximum when $\beta=2$ and $x=\infty$.

Since
$$a=l+0{\cdot}1 \quad . \quad . \quad \{\text{assumption (1)}\}$$
we get
$$\gamma=\frac{4\pi^2l^2\beta}{3\tau^2\sigma\sqrt{\mathrm{N}\frac{\pi}{\rho}\frac{1}{l+0{\cdot}1}}}\tan\theta \quad \text{from (12) and (15)}$$

$$=\frac{4\pi^2l^2\beta}{3\tau^2\sigma}\sqrt{\frac{\rho(l+0{\cdot}1)}{\mathrm{N}\pi}}\tan\theta \quad . \quad . \quad . \quad . \quad (16)$$

Now for 1 division deflexion at a scale distance of 1000 divisions, $\theta=\frac{1}{2000}$, so taking

$\tau=10$ seconds,
$\rho=1{\cdot}64\times10^{-6}$, and
$\sigma=50$,

corresponding with $\mathrm{B}=5000$ approximately, a high value for short permanent magnets, we get

$$\text{Current per 1 division} = \frac{6{\cdot}32}{10^{10}}l^2\sqrt{l+0{\cdot}1}\ \text{C.G.S. units},$$

$$=\frac{6{\cdot}32}{10^9}l^2\sqrt{l+0{\cdot}1}\ \text{amperes} \quad . \quad (17)$$

From this formula the following table has been calculated.

TABLE VII.

Half length of needles (l) in centimetres.	Amperes per division.	Divisions per Microampere.
0·05	$6{\cdot}12 \times 10^{-12}$	163,000
0·1	$2{\cdot}86 \times 10^{-11}$	35,000
0·15	$7{\cdot}16 \times 10^{-11}$	14,000
0·2	$1{\cdot}38 \times 10^{-10}$	7,250
0·3	$3{\cdot}60 \times 10^{-10}$	2,775
0·4	$7{\cdot}14 \times 10^{-10}$	1,400
0·5	$1{\cdot}22 \times 10^{-9}$	820

With these results before us it is interesting to examine how nearly the limits here given have been approached in actual instruments.

One galvanometer for which the dimensions of the magnets are known is that mentioned in line 9 of Table II. In this instrument the magnets are 0·82 centimetre long, therefore $l=0{\cdot}41$. From Table VII. an instrument with such a needle and wound to have a resistance of 1 ohm should give about 1350 divisions per microampere under the assumed conditions. On referring to Table II., line 9, we find the value of $D/r^{\frac{2}{5}}$ (the expression most suited for a high-resistance instrument with small coils) actually obtained to be 380, or little more than a quarter $\left(\frac{1}{3{\cdot}56}\right)$ the calculated possible value.

Another comparison can be made in the case of Mr. McKittrick's oscillograph, which has a needle of almost microscopic dimensions. In this instrument $l=0{\cdot}053$, and we see from Table VII., that we might possibly get 150,000 divisions per microampere when the resistance is one ohm, period 10 seconds, and scale-distance equals 1000 scale-divisions. Line 4 of Table II. gives the values 60,000 and 105,000 for $D/r^{\frac{1}{5}}$ and $D/r^{\frac{2}{5}}$ respectively. These numbers approximate more closely to the theoretical values than those for the Mudford galvanometer previously referred to, the probable reason being that the needle of Mr. McKittrick's

instrument consists of a piece of soft iron placed in a very strong magnetic field. The specific magnetism would therefore be much greater than that of short permanent magnets in a weak field such as used in controlling the needle of the Mudford galvanometer. In all probability the value of σ for Mr. McKittrick's instrument would be two or three times as great as the value used in calculating Table VII.

Section C.

Long versus *Short Period Galvanometers for Zero Methods.*

For rapidity and accuracy of working a short-period instrument is certainly better than a long-period one, providing its sensitiveness is as great. It is interesting, however, to inquire whether, with the same galvanometer, it is more expeditious to use a strong or a weak control, provided the sensitiveness when strongly controlled is sufficient for the purpose. For although the spot will move to its maximum elongation, corresponding with a given want of balance, more quickly when strongly controlled, that elongation will be less than when a weak control is used. Hence it is possible that a given small displacement of the spot from the zero position may occur in a shorter time when the control is weak than when it is strong, because such displacement is then a smaller fraction of the whole.

First applying general reasoning to the problem, we may notice that the deflecting couple produced by a given want of balance is initially independent of the control, and as the inertia of the moving system is constant, the acceleration at the beginning of the motion is the same whatever the control. If f be this acceleration the space traversed by the spot in a short time t is $\frac{1}{2}ft^2$, and this is therefore independent of the control, consequently for small displacements strong or weak control makes no difference in the time it takes to see that a want of balance exists.

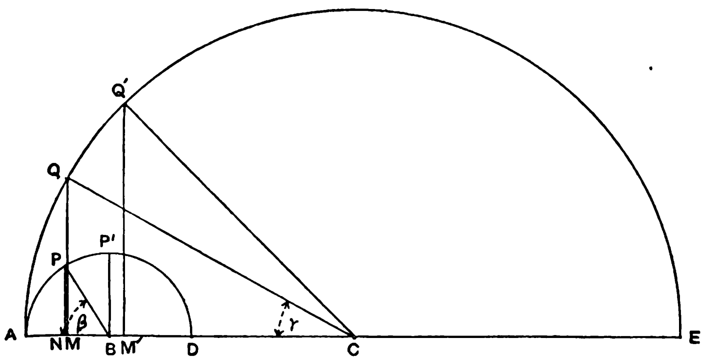
It is to be noted, however, that the weakly controlled system will take longer to return to zero, so that a longer wait is necessary before another test can be made. On the other hand, should the sensitiveness of the strongly controlled instrument be only just sufficient to get the desired accuracy,

then half a period must elapse before the want of balance can be detected, whereas the weakly controlled instrument would have moved over a greater distance in that time. Taking the controls in the ratio of 4 to 1, and therefore periods as 1 to 2, the movement of the weakly controlled instrument will be double that of the other in the case here considered. In this instance therefore the long periodic time is advantageous because it will show a very small want of balance more quickly, but the advantage is somewhat discounted by the longer time occupied in the return to zero.

Undamped Motion.

To consider the matter in further detail let us first take the case of undamped simple harmonic motion. Assuming the ratio of the controls to be as m^2 to 1, the periods will be in the ratio of 1 to m.

Fig. 2.



Draw two circles touching at A, centres at B and C, and of radii in the ratio of 1 to m^2. Draw lines BP and CQ making angles β and γ with AC and such that $\beta = m\gamma$. The distances from A of N and M, the projections on AC of P and Q respectively, will then represent the displacements from zero after a given time of the short and long period galvanometers respectively. From the figure (to scale) it will be seen that these are nearly equal when the time is, say, one-sixth of a period of the quick-moving system. And even when the time is one-fourth of the period of the strongly controlled

instrument, the difference in the displacements is not very marked, for in that time the short-period needle will have moved over the space AB, whilst the long-period needle will have traversed the distance AM′.

Again,

$$\begin{aligned} AN &= AB(1-\cos\beta), \\ &= b(1-\cos\beta) \quad \text{(say)}, \end{aligned}$$

and

$$\begin{aligned} AM &= AC(1-\cos\gamma), \\ &= c(1-\cos\gamma) \quad \text{(say)}\ ; \end{aligned}$$

but

$$c = m^2 b,$$

and

$$\beta = m\gamma,$$

$$\therefore \quad AN = b(1-\cos m\gamma),$$

and

$$AM = m^2 b(1-\cos\gamma).$$

$$\therefore \quad \frac{AM}{AN} = \frac{m^2(1-\cos\gamma)}{1-\cos m\gamma}. \quad \ldots\ldots \quad (1)$$

Expanding the cosines we get

$$\frac{AM}{AN} = \frac{m^2\left\{\frac{\gamma^2}{\underline{|2}} - \frac{\gamma^4}{\underline{|4}} + \frac{\gamma^6}{\underline{|6}} - \&c.\right\}}{\left\{\frac{m^2\gamma^2}{\underline{|2}} - \frac{m^4\gamma^4}{\underline{|4}} + \frac{m^6\gamma^6}{\underline{|6}} - \&c.\right\}}.$$

Dividing top and bottom by $\frac{m^2\gamma^2}{\underline{|2}}$, we have

$$\frac{AM}{AN} = \frac{1 - \frac{\gamma^2}{12} + \frac{\gamma^4}{360} - \&c.}{1 - \frac{m^2\gamma^2}{12} + \frac{m^4\gamma^4}{360} - \&c.}. \quad \ldots \quad (2)$$

This, to the 1st degree of approximation, is unity, and only differs slightly from unity when $m\gamma$ is considerable, thus confirming the general reasoning above. Taking m equal 2, as in the figure, and γ equal to $\frac{1}{2}$, *i. e.* $28^\circ \cdot 6$, we have

$$\begin{aligned} \frac{AM}{AN} &= \frac{1 - \frac{1}{48} + \frac{1}{5760} - \&c.}{1 - \frac{1}{12} + \frac{1}{360} - \&c.} \\ &= 1\cdot 06. \end{aligned}$$

Consequently the displacements differ by about 6 per cent. in a time equal to about one-sixth $\left(\frac{1}{2\pi}\right)$ of the period of the quick-moving instrument.

To put the matter in another way, we may determine the relative times after which the spots will have moved appreciably from the zero. A little consideration will show that these times will depend on the relation of the least visible movement to the maximum displacement produced by the want of balance it is necessary to detect. For example, if the sensitiveness of the short-period instrument is such as to give a fairly large movement of the spot for the given want of balance, then a perceptible movement will occur in a time which is a small fraction of its period, and this motion will take place in practically the same time as an equal motion of the spot of the long-period instrument. On the other hand, if the sensitiveness of the short-period instrument is only just sufficient to show the given want of balance when the spot reaches its maximum elongation, then half a period must elapse before the want of balance would be detected, and the more sensitive long-period instrument would have moved an equal distance in a shorter time. In this case, therefore, the more sensitive instrument would have a distinct advantage in quickness in showing the want of balance, but more time would be required for its return to zero before another test could be made.

To show the relative times under different conditions of required accuracy, and of different relative periods of similar galvanometers, we have calculated the following table.

The first column gives a measure of the sensitiveness of the short-period instrument in terms of the minimum possible value consistent with the desired accuracy, *e.g.*, the 1 in the first column indicates that the instrument is *only just sensitive enough* to indicate the maximum permissible error in balance; the 2 in the first column denotes that the instrument is twice as sensitive as absolutely necessary; and the 3 that it has three times the necessary sensibility &c. The numbers in line 1 below the various values of m^2 show the relative times required for the spot to move perceptibly, the time occupied by the slow-moving instrument in moving over the minimum

TABLE VIII.

Ratio of maximum displacement of short-period needle to least movement visible.	Ratio of $\frac{\text{Time taken by short-period galv. to move visibly}}{\text{,, ,, longer ,, ,, ,,}}$ when the sensitiveness of the longer-period instrument is m^2 times as great as that of the short-period instrument.				
	$m^2=2.$	$m^2=4.$	$m^2=16.$	$m^2=64.$	$m^2=10{,}000.$
1	1·41	1·50	1·55	1·60	1·63
2	1·06	1·09	1·10	1·10	1·11
3	1·034	1·05	1·053	1·054	1·055
4	1·025	1·04	1·045	1·047	1·048
8	1 011	...	...	...	1·021
10	1·01	...	...	...	1·019
20	1·005	...	...	...	1·007

perceptible distance being taken as unity. In column 2 line 1 we find the number 1·41, and this indicates that if the short-period instrument is only just sensitive enough to show the maximum permissible want of balance, then, doubling the sensitiveness by making the period $\sqrt{2}$ times as great, would enable the want of balance to be detected in a shorter time, the times being in the ratio of 1 to 1·41. Similarly for column 3 line 1, increasing the period to twice that of the short-period instrument and therefore increasing the sensitiveness four times would reduce the relative times to the ratio of 1 to 1·5. When m equals 100 (or the sensitiveness is increased ten thousand times, if such an increase were possible) the relative times become 1 to 1·63, showing a comparatively small extra diminution in time required to detect a want of balance, although the sensibility is very largely increased.

The numbers in line 2 show that when the short-period instrument is twice as sensitive as is absolutely necessary the gain in time required to detect the want of balance becomes less, the ratio of the times being 1 : 1·06 when m^2 equals 2, and only reducing to a value 1 : 1·11 when m^2 equals 10,000. Subsequent lines show that when the short-period

galvanometer is several times more sensitive than is absolutely necessary to obtain the desired accuracy, the gain in quickness in detecting a small error by weakening the control becomes unimportant, and so the small gain will be more than neutralized by the longer time taken in the return to zero.

Aperiodic Motion.

If the motion of the suspended system be aperiodic, as in well-damped d'Arsonvals, the problem is rather more complicated.

When damping forces act on a vibrating system the equation of motion is

$$\frac{d^2\theta}{dt^2}+\frac{2B}{I}\frac{d\theta}{dt}+\frac{C}{I}\theta=0,$$

$$\therefore \quad \theta=A_1\,\epsilon^{-(p+q)t}+A_2\epsilon^{-(p-q)t},$$

where

$$p=\frac{B}{I}, \text{ and } q^2=\frac{B^2}{I^2}-\frac{C}{I},$$

$$\text{or} \quad q^2=p^2-n^2, \text{ where } n^2=\frac{C}{I}.$$

Let $\qquad \theta=\alpha \quad \text{when} \quad t=0.$

Also $\qquad \dfrac{d\theta}{dt}=0 \quad ,, \quad t=0,$

then

$$\theta=\frac{\alpha}{2q}\epsilon^{-pt}\{(p+q)\epsilon^{qt}-(p-q)\epsilon^{-qt}\}.$$

If we now lessen the control to $\frac{1}{m^2}$ its initial value, so that the period, if undamped, would be lengthened m times, the sensitiveness will increased m^2 times, and consequently the value of α is increased m^2 times. This change of C alters q, but p is not affected thereby. Let the new value of q be q'.

Under the altered conditions we have

$$\theta'=\frac{\alpha'}{2q'}\epsilon^{-pt}\{(p+q')\epsilon^{q't}-(p-q')\epsilon^{-q't}\}.$$

We may also write θ in the form

$$\theta=\frac{\alpha}{2q}\{(p+q)\,\epsilon^{-(p-q)t}-(p-q)\,\epsilon^{-(p+q)t}\},$$

and expanding

$$\theta = \frac{\alpha}{2q}\{(p+q)[1-(p-q)t + \frac{(p-q)^2 t^2}{\underline{|2}} - \&c.]$$

$$-(p-q)[1-(p+q)t + (p+q)^2 t^2 - \&c.]\}$$

$$= \frac{\alpha}{2q}\{2q + (p^2-q^2)\frac{t^2}{\underline{|2}}[p-q-(p+q)]$$

$$-(p^2-q^2)\frac{t^3}{\underline{|3}}[(p-q)^2-(p+q)^2] + \&c.\}$$

$$= \frac{\alpha}{2q}\{2q + (p^2-q^2)\frac{t^2}{\underline{|2}}(-2q) - (p^2-q^2)\frac{t^3}{\underline{|3}}(-4pq) + \&c.\}$$

$$= \alpha\{1-(p^2-q^2)\Big[\frac{t^2}{\underline{|2}} - 2p\frac{t^3}{\underline{|3}} + \text{terms involving } t^4 \text{ \&c. } p \text{ and } q\Big]$$

$$= \alpha\{1-n^2\Big[\frac{t^2}{\underline{|2}} - 2p\frac{t^3}{\underline{|3}} + \text{terms \&c.}\Big]$$

$$\therefore \theta - \alpha = -\alpha n^2\Big[\frac{t^2}{\underline{|2}} - 2p\frac{t^3}{\underline{|3}} + \text{terms \&c.}\Big].$$

Similarly

$$\theta' - \alpha' = -\alpha' n'^2\Big[\frac{t^2}{\underline{|2}} - 2p\frac{t^3}{\underline{|3}} + \text{terms involving } t^4 \text{ \&c. } p \text{ and } q'\Big].$$

Hence the ratio of the two displacements is –

$$\frac{\theta'-\alpha'}{\theta-\alpha} = \frac{\alpha' n'^2}{\alpha n^2}\frac{\Big[\frac{t^2}{\underline{|2}} - 2p\frac{t^3}{\underline{|3}} + \text{terms in } t^4 \text{ \&c. } p \text{ and } q'\Big]}{\Big[\frac{t^2}{\underline{|2}} - 2p\frac{t^3}{\underline{|3}} + \text{terms in } t^4 \text{ \&c. } p \text{ and } q\Big]},$$

but $\alpha' = m^2\alpha,$

and $n'^2 = \frac{1}{m^2}n^2,$

$$\therefore \quad \alpha' n'^2 = \alpha n^2.$$

$$\therefore \quad \frac{\theta'-\alpha'}{\theta-\alpha} = 1 \times \frac{[\qquad]}{[\qquad]}.$$

Thus, when $\frac{t^4}{\underline{|4}}$ and higher powers are small, the ratio is unity, consequently there is no difference in the rapidity of detecting want of balance when the sensitiveness of the short-period instrument is sufficient for the purposes of the experiment.

To find the ratio in the other cases we may observe that the general term in t^r is, when r is odd,

$$-\frac{t^r}{\underline{|r}}\left\{(r-1)p^{r-2}+\frac{(r-1)(r-2)(r-3)}{\underline{|3}}p^{r-4}q^2+\ldots+(r-1)pq^{r-3}\right\};$$

and when r is even

$$+\frac{t^r}{\underline{|r}}\left\{(r-1)p^{r-2}+\frac{(r-1)(r-2)(r-3)}{\underline{|3}}p^{r-4}q^2+\ \ldots\ +q^{r-2}\right\}.$$

Decreasing the control decreases n^2 and therefore increases q,

$$\because \qquad q^2=p^2-n^2.$$

Hence $\qquad q'$ is $> q$,

consequently the term in t^4 in the numerator of $\frac{\theta'-\alpha'}{\theta-\alpha}$ will be greater than the corresponding term in the denominator, and this indicates that the displacement of the slow-moving instrument gains on that of the quick-moving one as the time increases.

The above investigation teaches us that when we have a galvanometer whose control can be readily altered and whose sensitiveness can be easily made ample for the purpose of the test, then *for rapid working adjust the control so that the sensitiveness is, say, two or three times as great as is absolutely necessary for the desired accuracy.*

Discussion.

Prof. Threlfall thought the authors' method of comparing galvanometers very misleading. The results obtained in their comparison of the oscillograph (3,310,000) and the suspended-coil galvanometer (27) might be regarded as the *reductio ad absurdum* of the proposed system. The absurdity arose from the dissimilarity of the two instruments. Moreover, the proposed system ignored the fact that sensitiveness may be obtained by optical as well as by electromagnetic means. Optical sensitiveness, owing to its greater stability, was to be preferred to electromagnetic sensitiveness. The fundamental problem in the construction of galvanometers is an optical one; it is necessary to decide the mass and dimensions of the suspended parts so as to ensure (1) optical accuracy, and (2) electromagnetic sensitiveness. Thus, to some extent, the weight of the mirror determines the thickness of the suspension. As an instance of what might be done by optical methods, Prof. Threlfall referred to work done by himself and Mr. Brearley (Phil. Mag. 1896), in which it was possible to measure to $1{\cdot}48 \times 10^{-13}$ amperes, and, with special refinements, to 3×10^{-14} amperes. He had found that the best diameter for glass mirrors was 1·1 centim., with a weight just under 0·5 grm. These were used with a scale at 276 centim., read by a microscope to 0·04 millim. The course of the light was:—Lamp, large lens, small scale, mirror, eyepiece. The period was 25 secs., and the resistance 50,000 ohms. Even better results could be obtained by using mirrors of quartz or of blood-stone. Quartz is incomparably to be preferred to glass. Such figures indicated what could be done by optical sensitiveness—the sensitiveness that the authors ignored. It was pointed out by Prof. Threlfall that the controlling field for galvanometers of the "Thomson" type should be uniform. This was best secured by using two magnets, one above, and one below the needles.

Prof. Perry said the authors had not asserted that a galvanometer with higher figure of merit, according to their classification, was *superior* to another of lower figure. It must be agreed that the figure they obtain is a very

valuable datum for the comparison of instruments designed for similar purposes, for instance in classifying those used by Prof. Threlfall. Mr. Duddell was to be congratulated on the extreme sensitiveness and small period of his oscillograph.

Prof. AYRTON, referring to Prof. Threlfall's *reductio ad absurdum*, admitted that the criticism would carry some conviction if the two instruments were of different kinds; if, for instance, one possessed a suspended needle and the other a suspended coil. But the argument failed, because both instruments were of the suspended-coil type. In one of them Mr. Duddell had developed the advantages to be gained by reducing the air-gap. To form an opinion of electromagnetic improvements in galvanometers, it was necessary to reduce the results of all instruments to some system of classification. There was no objection, after that, to adding a good mirror, and reading by a good microscope.

ERRATUM.

On page 1 of this volume the title of the paper by Prof. Stroud should read:

"*A Telemetrical Spherometer and Focometer.*"

XXIII. *A Simple Method of Reducing Prismatic Spectra.*

By EDWIN EDSER, *A.R.C.S., and* C. P. BUTLER, *A.R.C.S.**

IN order to determine, from spectroscopic measurements, the wave-lengths corresponding to the bright lines in a prismatic spectrum of a metal or gas, one or other of the following methods is generally used. From preliminary measurements made of the deviation corresponding to a number of known lines in the solar spectrum, or the line spectrum of some metal or gas, a curve is drawn giving the relation between deviation and wave-length. Owing to the necessity of determining a very large number of points on this curve in order to render its form trustworthy, this operation is a very tedious one, and to an observer insufficiently acquainted with the reference spectrum involves great difficulty and uncertainty. This curve, however, having been drawn, the wave-length of any line in another spectrum obtained with the same spectrometer (no alteration of the adjustments having been made) can be immediately determined from a measurement of its deviation. On the other hand, where photographs of spectra are employed the most usual practice at present is to photograph a reference solar-spectrum alongside the one under examination. To an observer of sufficient experience it is possible to identify any of the numerous Fraunhofer lines with the corresponding lines in a Rowland's standard map; and thus the wave-length of any line in the unknown spectrum may be determined by inspection. In spite of the perfection attainable by the above methods when employed by a trained observer, it has appeared to us that a simpler one, capable of giving accurate results in the hands of an experimenter without special experience in spectroscopy, might often be found of some value. The production of interference-bands in a continuous spectrum seemed capable of furnishing a reference spectrum which could be advantageously employed for this purpose, most of the difficulties incident to the above-mentioned methods being

* Read May 27, 1898.

entirely eliminated. We have, therefore, devoted some time to the examination of various methods by which such interference-bands might be produced, with the object of selecting the simplest, and determining the degree of accuracy finally attainable by its employment. The results of our work in this direction we beg to lay before the Society this evening.

If the image of a system of rectilinear interference-fringes be formed in the plane of, and parallel to, the collimator-slit of a spectrometer so that only a small part of the breadth of one band falls on the slit, the resulting spectrum will be crossed by vertical black bands, varying in number and breadth with the order of the interference-band from which the spectrum is derived *. If the interference-fringes are displaced across the slit the black bands in the spectrum will become finer and more numerous as the central interference-fringe recedes from the slit. Even when the coloured fringes have become invisible at the position of the slit owing to the high relative retardation of the interfering pencils, the bands in the spectrum remain quite distinct, becoming indistinguishable only when so fine that the resolving and dispersive powers of the spectroscope are insufficient to separate them.

In our earlier experiments we focussed the image of an air-film, contained between two plane and parallel glass surfaces, on the slit of the spectrometer †. The image should be obtained by means of light reflected from the film, the spectrum bands obtained when transmitted light is used being very faint. This method, which is theoretically the most perfect, has the disadvantage that a somewhat careful adjustment is necessary in order to insure good results. We have therefore sought for some simpler method. It is unnecessary here to detail the various methods which we have successively tried; it will suffice to describe the arrangement ultimately adopted as being the simplest, whilst complying

* Fizeau and Foucault, *Ann. de Chim. et de Phys.* 3rd series, tom. xxvi. p. 138 (1849); *Comptes Rendus*, Nov. 24, 1845.

† An air-film has been used somewhat similarly by Rubens in order to calibrate a prism for infra-red light. Wied. *Ann.* vol. xlv. (1892) p. 238.

sufficiently closely with the ideal conditions to insure trustworthy results.

Let us suppose that a transparent parallel-sided film of thickness d is placed immediately against the slit of the spectroscope and illumined with white light. Owing to the interference of the ray directly transmitted and that twice internally reflected within the film, there will be bright bands in the spectrum separated by darker intervals, the wave-lengths $\lambda_0, \lambda_1, \lambda_2, \ldots \lambda_r, \ldots \lambda_m$ corresponding to the bright bands being given by the equations

$$2\mu d = n\lambda_0 = (n+1)\lambda_1 = (n+2)\lambda_2 = \ldots = (n+r)\lambda_r = (n+m)\lambda_m,$$

where μ is the refractive index (supposed independent of the wave-length) of the substance of the film, whilst n may be any integral number.

If λ_0 and λ_m are known n can be determined from the equation

$$n = \frac{m\lambda_m}{\lambda_0 - \lambda_m}, \quad \ldots\ldots \quad (1)$$

where the interference-band at λ_m is the mth from that at λ_0; λ_m being towards the violet, and λ_0 towards the red end of the spectrum.

The wave-length λ_r corresponding to any other interference-band (the rth from that at λ_0) is now immediately given by

$$\lambda_r = \frac{n\lambda_0}{n+r}. \quad \ldots\ldots \quad (2)$$

It is therefore possible, by means of a series of interference-bands produced in the spectrum in the above manner, to calculate the wave-length corresponding to any part of the spectrum, having given any two lines of known wave-lengths sufficiently remote from each other.

Of course it is impossible to obtain a solid film of any substance whose dispersion is sufficiently small to render the above reasoning even approximately correct. Recourse must then be had to an air-film between two transparent plates. Since the film can now no longer be placed immediately against the collimator slit, some indefiniteness of the interference-

bands will result; but if the plate next to the slit is not more than 3 millim. in thickness no trouble will arise from this cause, at any rate with a spectrometer whose collimator-tube is more than a foot in length.

A very considerable improvement in the interference-bands thus produced may be effected by partially silvering the two surfaces enclosing the air-film. In the first place the contrast between the bright and dark bands is considerably enhanced; indeed, if both surfaces be silvered so as to reflect about 75 per cent. of the incident light, the dark spectrum bands become almost black. The thicker the silver is the greater will be the contrast, the only limit being prescribed by the diminution of the total light transmitted.

Another important advantage gained by silvering the surfaces is the much sharper definition of the resulting bands. Messieurs C. Fabry and A. Perot* have pointed out that when monochromatic light is transmitted through a film enclosed between two plane and approximately parallel silvered surfaces, the resulting interference-bands present the appearance of sharp well-defined bright lines separated by broad black intervals. The explanation of this interesting phenomenon is quite simple. Let the real part of $\epsilon^{\iota p(x-\mathrm{V}t)}$ be the equation of the incident wave, whilst a and b are the respective coefficients of reflexion and transmission at the silver surface. Since Wiener has shown that the phase change for light reflected normally at a silver surface in air is very approximately equal to half a wave, a may be taken as wholly real.

Consequently, if the thickness of the air-film is d, the resultant transmitted beam on emergence will be given by the real part of the sum of the infinite series

$$b^2\epsilon^{\iota p-\mathrm{V}t+d)}+b^2a^2\epsilon^{\iota p(x-\mathrm{V}t+3d)}+b^2a^4\epsilon^{\iota p(x-\mathrm{V}t+5d)}+\text{\&c.}$$

$$=b^2\epsilon^{\iota p(x-\mathrm{V}t+d)}\{1+a^2\epsilon^{2\iota pd}+a^4\epsilon^{4\iota pd}+\text{\&c.}\}$$

$$=b^2\epsilon^{\iota p(x-\mathrm{V}t+d)}\cdot\frac{1}{1-a^2\epsilon^{2\iota pd}}. \qquad (3)$$

Substituting $\cos 2pd+\iota\sin 2pd$ for $\epsilon^{2\iota pd}$, and rationalizing

* *Ann. Chim. Phys.* xii. pp. 459–501 (1897).

the denominator of (3), we find the transmitted wave to be equal to the real part of

$$A\epsilon^{\iota p(x-Vt+d+e)},$$

where

$$A^2=\frac{b^2}{1+a^4-2a^2\cos 2pd}, \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\tan pe=\frac{\sin 2pd}{1-a^2\cos 2pd}.$$

Now A^2 is proportional to the intensity of the transmitted light; hence as d is varied the intensity will vary from $I_{(\max.)}=\frac{b^2}{(1-a^2)^2}$ to $I_{(\min.)}=\frac{b^2}{(1+a^2)^2}$. Taking Michelson's expression for the visibility of interference bands, viz., $\frac{I_{(\max.)}-I_{(\min.)}}{I_{(\max.)}+I_{(\min.)}}$, we find that this becomes equal to $\frac{2a^2}{1+a^4}$, which will have a maximum value when $a=1$. Thus the visibility of the bands will increase with the reflecting power, and therefore with the thickness of the silver.

Also, from (4) we obtain

$$\frac{dI}{dd}=c\frac{-4pa^2b^2\sin 2pd}{(1+a^4-2a^2\cos 2pd)^2}.$$

Consequently I varies much more rapidly as d is increased when $2pd$ is nearly equal to $2n\pi$ than it does when $2pd$ is nearly equal to $(2n+1)\pi$. The bright bands will therefore be very narrow and sharply defined, separated by broad intervals very nearly black *.

It may be useful here to describe in detail the exact method of procedure finally adopted. It has not been found necessary to use optically worked glass; good ordinary plate-glass gives perfect results. Sextant glasses have been recommended to us for this purpose. It is well to select two plates having

* A similar result has been noticed in connexion with the interference of electrical waves. See "Electrical Interference Phenomena, somewhat analogous to Newton's Rings, but exhibited by waves passing along wires of which a part differs from the rest," by E. H. Barton, D.Sc. Proc. Roy. Soc. vol. liv. p. 85.

the most suitable surfaces. This can be done by placing one plate on another, the two adjacent surfaces having previously been cleaned with cotton-wool, and viewing the air-film between them by reflected light from a sodium flame. The bands formed when the plates have been gently pressed together should be nearly straight and each one at least 2 millim. or 3 millim. in breadth.

The two selected surfaces should now be silvered somewhat heavily. The milk-sugar process*, in which the silver is deposited very slowly, has been found to give good results.

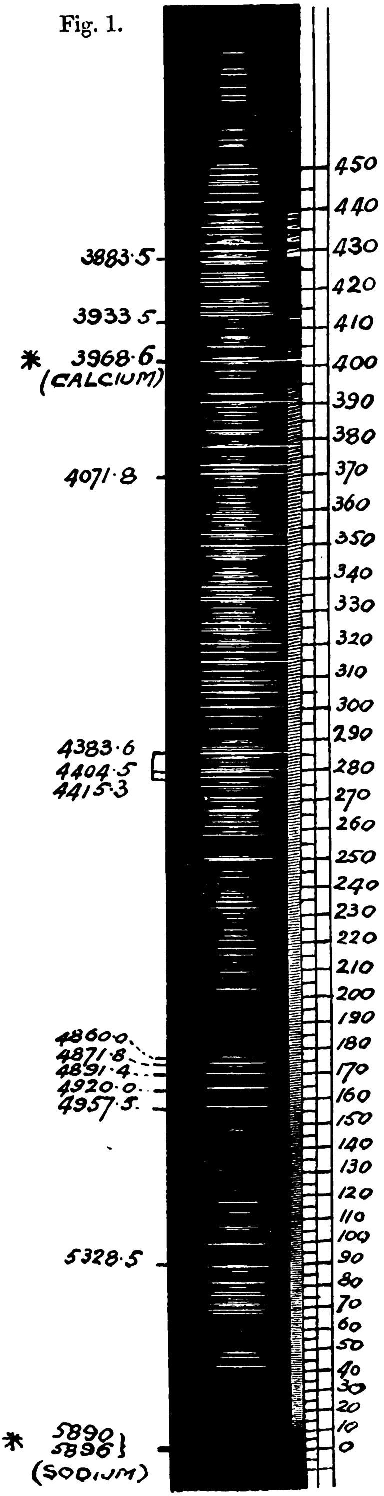
A simple mechanical arrangement for adjusting the two silvered surfaces for parallelism, the distance between them being also capable of adjustment, could easily be designed. We have found, however, that if a little soft wax be placed round the edges of the plates a perfect adjustment can be obtained by simply pressing the plates together with the fingers. The photograph accompanying this paper was obtained using this arrangement. To adjust for parallelism, view a spot of light, or the filament of an incandescent electric lamp, through the silvered surfaces. A long train of images, due to multiple reflexions, will generally be visible. These images having been brought into coincidence, interference-bands will generally be seen on viewing a sodium flame through the silvered surfaces. These are adjusted, by pressure applied to the glass plates, to be as broad as possible. When the adjustment is nearly completed there is often some difficulty in seeing the bands, due to the fact that for a parallel air-film viewed normally the interference-bands are formed at an infinite distance in front of the film†. At this stage the bands should be viewed from as great a distance as possible. The perfection of the results finally obtained will depend greatly on the accuracy with which this adjustment is performed.

If the collimator-slit of the spectrometer be now illumined

* For the exact process employed by us see 'Nature,' Sept. 23, 1897. "On the Phase-change of Light when Reflected from a Silver Surface," by Edwin Edser and H. Stansfield.

† A. A. Michelson on "Interference Phenomena in a new form of Refractometer," Phil. Mag., April 1882

Fig. 1.



COMPARISON SCALE FOR REDUCTION OF SPECTRA.

by a slightly convergent beam from an arc-lamp, and the plates be placed in front of the slit, and as near to it as possible, the spectrum will be found to consist of bright lines separated by almost black intervals. The best results will of course be obtained when the plates are in such a position that the slit is parallel to the direction of the interference-bands seen with sodium light. The closeness of the bands will depend on the thickness of the air-film between the silvered surfaces. For photographic purposes we have adopted the plan of covering either the upper or lower half of the slit with a piece of black paper stuck on with soft red wax before placing the plates in position. The necessary exposure will vary from about half a minute to three minutes (using Edwards's snap-shot isochromatic plates), according to the nature of the spectrometer employed. It is well to introduce a little common salt into the arc while this exposure is being made, as thus the D lines, as well as the H and K lines, will be superimposed on the bands. Another piece of black paper having been placed so as just to cover the exposed half of the slit, the first piece is removed, and the spectrum which it is wished to examine is photographed. Fig. 1 (p. 213) is a specimen of an iron spectrum together with a reference interference-spectrum obtained in this way. It will be noticed that the interference-bands in the violet part of the spectrum are slightly inclined to the vertical. It is easy to adjust the glass plates so that this is not the case, but we have selected this photograph in order to point out that if readings are taken on the line of junction of the two spectra no error will result from such a want of adjustment.

Starting from the red end of the spectrum, every fifth and tenth band can be marked and the whole numbered. The following exhibits the procedure when the wave-lengths of only a few lines are required.

One or other of the D lines and the H or the K line will generally be found to form the best datum lines. In the present case the following datum lines were used:—

Scale No. 90·2.	Wave-length 5328·5 (λ_0)
„ 402 3.	„ 3968·6 (λ_m) (H).

Then according to equation (1)

$$n=\frac{m\lambda_m}{\lambda_0-\lambda_m},$$

$$m=402{\cdot}3-90{\cdot}2=312{\cdot}1,$$

$$\lambda_0-\lambda_m=1359{\cdot}9\,;$$

hence

$$n=910{\cdot}8.$$

To find the wave-length of the line whose scale-number $=371{\cdot}2$:

$$r=371{\cdot}2-90{\cdot}2=281,$$

$$\lambda_r=\frac{n\lambda_0}{n+r}=\frac{910{\cdot}8\times 5328{\cdot}5}{910{\cdot}8+281}$$

$$=4072{\cdot}2.$$

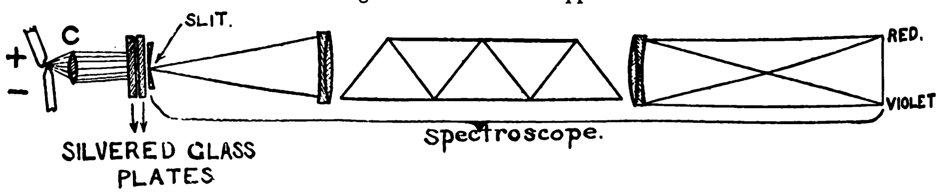
The true value of this wave-length is 4071·8, giving an error of +·4 tenth-metres.

The following Table shows the calculated and true values for a number of lines in the above spectrum. It is given in order to indicate the degree of accuracy attainable. It is worth notice that these results were all obtained without the use of a travelling microscope, or in fact any auxiliary appliance other than an ordinary pocket-lens. With the latter it is easy to estimate the position of a line relatively to the interference-scale to within one-tenth of a band. Further, the interference-scale in the present instance was purposely made rather coarse so as to admit of reproduction. With a finer scale a greater degree of accuracy might be attained.

Scale No.	Wave-length (calculated).	True Wave-length.	Error.
			tenth-metres.
371·2	4072·2	4071·8	+·4
286·5	4383·7	4383·6	+·1
281·1	4405·2	4404·8	+·4
278·6	4415·2	4415·3	−·1
354	4131·8	4132·2	+·4
413·1	3933·9	3933·5 (K)	+·4

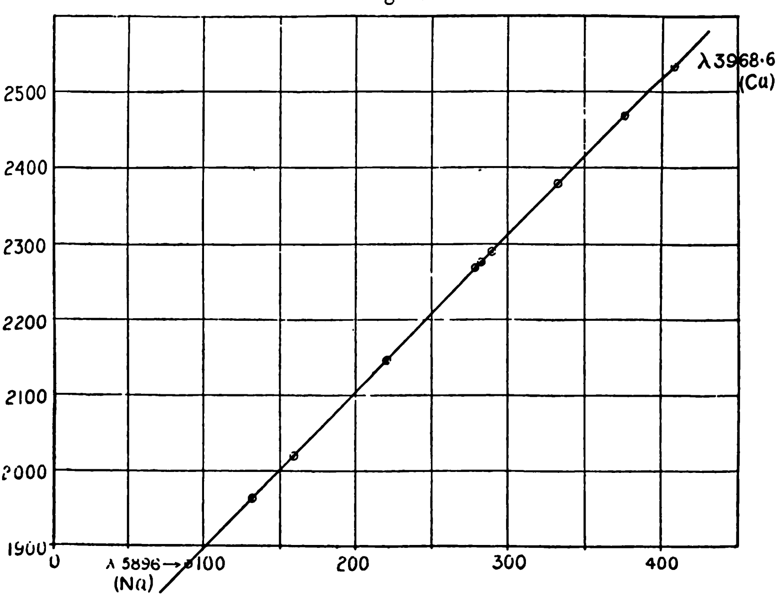
When it is required to determine the wave-length corresponding to a great number of spectral lines a graphical method may be employed. If we write $\frac{1}{\lambda}=\mathrm{L}=$ the frequency of the light vibrations, we obtain the simple relation $\frac{n+r}{\mathrm{L}}=$ constant, or $\mathrm{L}=\mathrm{K}(r+n)$, *i.e.* the relation between r and L may be expressed by a straight line.

Fig. 2.
Diagrammatic Plan of Apparatus.



Plotting the frequencies vertically, and r horizontally, we obtain fig. 3. It is only necessary to mark off the scale-

Fig. 3.



divisions, starting from zero, along the horizontal axis, and to mark off vertically above their respective scale-divisions

the frequencies of the two standard lines, joining the extremities of the latter by a straight line. The frequency corresponding to any given scale-number is read off directly. The accuracy attainable by this method, in which no calculations whatever are involved, is similar to that obtained by the method previously described.

It is, of course, unnecessary to take a separate photograph of the reference interference-scale for every spectrum to be examined. If the D lines are superposed on the original interference-scale, and occur also in every succeeding spectral photograph obtained, the reference scale can be photographed once for all, provided the adjustments of the spectrometer remain unaltered. The photographic scale can be placed with its film in contact with that of the photograph bearing the unknown spectrum, and the D lines having been brought into coincidence the procedure indicated above may be proceeded with.

For eye-observations the most convenient arrangement would be to place a small plate of optically worked glass between the reference prism, generally provided with a spectrometer, and the slit, a simple arrangement serving to adjust the adjacent surfaces (which should be silvered) for parallelism.

It will be seen that the phase-changes produced by the silver do not introduce any serious errors into the final results. Wiener* has shown that for light reflected from a silver film of sufficient thickness the phase-change is very nearly independent of the wave-length. To further test this a streak of silver was rubbed off the glass plate which is placed next to the collimator-slit, and a photograph of the spectral bands obtained. The displacement of the bands where the light had been reflected from the silver, relatively to the bands formed where the light had been reflected from the clear glass, was practically constant for the whole length of the spectrum.

In conclusion, we think that it may be claimed that by means of this application of a well-known principle to spectroscopy, the difficulties incident to the reduction of prismatic spectra in terms of wave-lengths or frequencies are greatly

* O. Wiener, Wied. *Ann.* xxxi. p. 629 (1887).

reduced, the whole process as above described requiring no special experience in the experimenter.

The experimental work incidental to this investigation has been performed partly at the Davy-Faraday Laboratory, Royal Institution, and partly at the Royal College of Science, South Kensington. For the facilities afforded us individually at these institutions our joint thanks are due.

Discussion.

Prof. Threlfall congratulated the authors on their discovery of a method that would greatly reduce the labour of calibrating spectra, and at the same time give such accurate results.

Prof. Boys said the simplicity of the apparatus added greatly to the value of the method. It would seem to him better if the slit were somehow contrived within the film-space. All want of definition due to rays falling at different angles upon the collimator object-glass would thus be avoided, and only a small part of the glass plates, *i.e.* the slit, would require to be strictly parallel planes. The limit of accuracy in the authors' method depended upon the collimator, not upon the optical perfection of the silvering of the plates.

Mr. Butler pointed out that previous methods had always required experienced spectroscopists for mapping out results. In the new method that work could easily be done by an assistant.

Mr. Edser said that, by putting the two plates immediately in front of the slit, only a very small part of the glass is concerned in the action; light coming through at an angle would not reach the lens in the collimator.

XXIV. *Attenuation of Electric Waves along a Line of Negligible Leakage.* By EDWIN H. BARTON, *D.Sc., F.R.S.E., Senior Lecturer in Physics, University College, Nottingham* *.

SHORTLY after the publication of my previous paper † on this subject, Mr. Oliver Heaviside pointed out to me that it would be interesting to compare the observed attenuation in the transit of the waves with that to be expected on the long-wave theory. He was also good enough to draw my attention to Lord Rayleigh's high-frequency formula for the effective resistance of wires to alternating currents as approximately applicable to the case. He feared, however, that the experimental value of the attenuation would be found considerably higher than the theoretical one due to resistance of the wires simply.

It immediately occurred to me that the extra attenuation observed over that theoretically expected might be due to the leakage conductance of the wood separators which were used at intervals along the line to keep the two parallel wires properly spaced. But at the time in question the line was already dismantled. It was therefore impossible to say precisely what the resistances of the separators were, nor how many of them were in use, when the attenuation was determined.

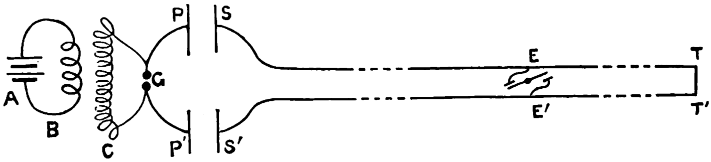
Assuming that the separators occurred on the average at a metre apart along the line, a resistance for each of the order half a megohm would suffice to reconcile the experiments with the theory referred to. On testing a few of the separators their resistance dry was found to be of the order ten megohms, but the resistance of one about a minute after momentary immersion in water had fallen to 0·2 of a megohm. It therefore seemed desirable to rig up a new line with sepa-

* Read June 10, 1898.

† Phil. Mag. August 1897; Proc. Phys. Soc. Dec. 1897, Jan. 1898.

rators of such high resistance as to render the leakage conductance of the line quite negligible. This was accordingly done and the attenuation again measured. The result, however, only confirmed the value previously obtained. This value is indeed higher than that predicted by the theory approximately applicable to the case. Still, upon careful examination of the differences between the experimental case and that contemplated in the theory, I think it will be conceded (1) that this difference of values is in the right direction, and (2) that it is not large enough to shake confidence in the substantial accuracy of the experiments. This is discussed in detail at the end of the paper.

Fig. 1.

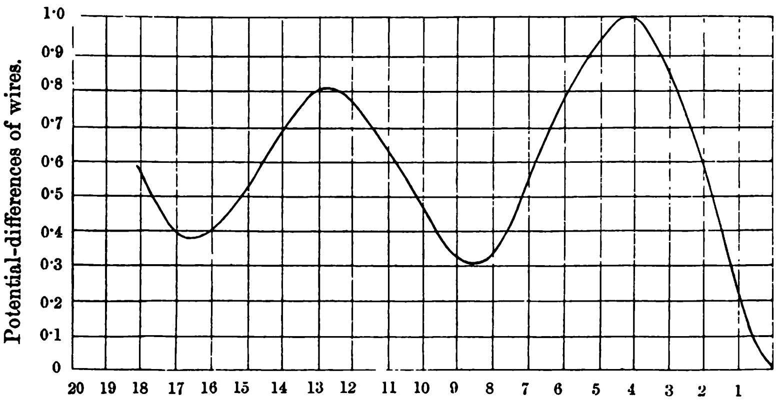


Experimental Arrangement.—The arrangement of the apparatus adopted in the determination is diagrammatically represented in fig. 1. In this figure, A denotes the battery of two storage-cells, B and C are the primary and secondary coils respectively of the induction-coil, of which C has a resistance of 3000 ohms and an inductance of about 20 henries. G is the spark-gap, which was adjusted to 2 millim. The wire PGP′, measured along the semicircle, was about 2 metres. PP′ are condenser-plates of zinc 40 centim. diam. placed opposite to and 30 centim. distant from the precisely similar plates S S′. The line is represented by S E T T′ E′ S′, and consists of two parallel copper wires 1·5 millim. diam., and kept 8 centim. apart by wood separators at intervals of about one metre. These wood separators are about 12 centim. long, 2·5 centim. wide, and 1·2 centim. thick, and were well saturated in melted paraffin-wax before use. Their resistance was thus raised to something over 60,000 megohms each, which rendered the leakage of the line quite negligible.

E E′ denotes the electrometer, which has a single plane needle, initially uncharged, and suspended by a fine quartz fibre between two disks attached to the line at E and E′. The needle is therefore electrified by induction whenever a wave passes E E′, and its ends are consequently attracted to the disks whatever the sign of their potential-difference. T T′ signify the two bridges alternately used at or near the end of the line, namely, (1) the bridge of critical resistance which absorbs the waves completely, and (2) the bridge which reflects them completely. The lengths of the line before and after the electrometer will be stated in connexion

Fig. 2.

Curve showing Wave-Train advancing to the right.



with each experiment. The waves generated by the primary oscillator were about 8·5 metres long, and, when on the line, are of the form represented in fig. 2.

The electric waves propagated along the line are thus seen to be in the form of a damped train with the large end leading; the tail after about ten or a dozen waves being almost negligible. The curve of fig. 2 was obtained experimentally by Bjerknes' method, and is identical with

that marked E in the paper on "Absorption of Electric Waves"*.

Method of Determination.—The method used for determining the attenuation of the waves in their transit along the line was the same as that described in the first paper already referred to. It consists essentially of alternately using at the end of the line a bridge which completely absorbs the waves and a bridge which completely reflects them. In the first case the wave-train goes but once past the electrometer, and gives a deflexion δ_1, say; in the second case the electrometer is affected by the sum to infinity of two geometrical progressions due to the forward and return trains respectively. Let the consequent deflexion be δ_2. Then the ratio r of δ_2 to δ_1 is a function of the attenuation factor $e^{-\sigma x}$, the reflexion coefficient at the oscillator, ρ, and the lengths of the line, l_1 and l_2, before and after the electrometer respectively. It is convenient, however, to solve the equations first for ρ^2 and s, where $10^{-sx'}=e^{-\sigma x}$, x' denoting lengths in metres and x lengths in centim. along the line.

Observations and Results.—The present determination is based upon observations with lengths of 116 metres and 65 metres respectively before the electrometer, and 48 metres in each case after. To determine the ratio δ_2/δ_1 in the second case 41 electrometer-throws were sufficient. These are given in Table I. In the first case, as the sparks of the oscillator were less regular, over a hundred electrometer-throws were taken before the required accuracy was obtained. Table II. summarizes the data and results.

* Phil. Mag. January 1897, p. 43; Proc. Phys. Soc. vol. xv. p. 27

TABLE 1.

Showing electrometer readings for $l_1=65$ m., $l_2=48$ m.

Electrometer-throws with Absorbing Bridge only at end of Line.		Electrometer-throws with Reflecting Bridge		Ratios of Throws, viz.:—	
Actual observations.	Interpolated means.	at 48 m. $-\lambda/8$ beyond the Electrometer.	at 48 m. $+\lambda/8$ beyond the Electrometer.	$\frac{\text{col. 3}}{\text{col. 2}}$.	$\frac{\text{col. 4}}{\text{col. 2}}$.
20					
	21·5	66	...	3·07	
23					
	23	...	63	...	2·74
23					
	23·5	80	...	3·40	
24					
	24·5	...	77	...	3·14
25					
	25·5	80	...	3·14	
26					
	27	...	75	...	2·78
28					
	27	82	...	3·04	
26					
	24	...	72	...	3·00
22					
	24·5	79	...	3·22	
27					
	27	...	71	...	2·63
27					
	27	78	...	2·89	
27					
	26·5	...	72	...	2·72
26					
	27	81	...	3·00	
28					
	27	...	77	...	2·85
26					
	26·5	76	...	2·87	
27					
	27	...	78	...	2·89
27					
	27·5	78	...	2·84	
28					
	26·5	...	76	...	2·87
25					
	26	79	...	3·04	
27					
	24·5	...	72	...	2·94
22					
		Whence Final Mean Ratio = 2·955 ± 0·0254			

Table II.

Summary of Data and Results of Experiments, with previous result added for comparison.

	Lengths of Line: before Electrometer $=l_1$.	Lengths of Line: after Electrometer $=l_2$.	Ratios of Throws $=r$.	The unknowns for which the equations were solved: s.	The unknowns for which the equations were solved: ρ^2.	Attenuation Constant, σ.	Reflexion Coefficient at Oscillator, ρ.
	metres.	metres.					
Present Determination.	116	48	$2{\cdot}555 \pm 0{\cdot}044$	0·000566	0·513	0·0000130	0·716
	65	48	$2{\cdot}955 \pm 0{\cdot}0254$				
Best of Previous Determinations.	117·5	20	$2{\cdot}41 \pm 0{\cdot}04$	0·000564	0·4776	0·0000130	0·69
	65·0	20	$2{\cdot}744 \pm 0{\cdot}035$				

Discussion of Results.—Heaviside has shown* that for electric waves along a pair of parallel leads the attenuation factor is

$$e^{-(R'/2Lv+K/2Sv)x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R′, K, L, and S are respectively the effective resistance to the waves in use, the leakage conductance, the inductance, and the permittance, all per unit length of the line, v is the speed of light, x the space coordinate along the line, all units being on the C.G.S. electromagnetic system.

This, for the present case, reduces to

$$e^{-R'x/2Lv}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

since $K/2Sv$ is of the order one hundred-thousandth of $R'/2Lv$.

Now to obtain an approximate value of R′, use Lord Rayleigh's high-frequency formula †

$$R' = R\sqrt{\tfrac{1}{2}pa\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* 'Electrical Papers,' vol. ii. p. 148, and elsewhere.

† Phil. Mag. May 1886, p. 390, equation (26).

where R is the resistance to steady currents of a length l of the wire, $\alpha = l/\mathrm{R} =$ the conductivity per unit length of the wire, $p = 2\pi$ times the frequency of the waves, and μ is the magnetic permeability of the wires.

In the experiments under discussion the numerical data were as follows:—

$\mathrm{R} = 1{\cdot}1 \times 10^5$ C.G.S. units per centim. (ascertained by P.O. Box tests, 10 metres of wire being 0·11 ohm).

$p = 2\pi \times 35 \times 10^6$ per second (deduced from λ and v).

$\mu = 1$ for copper wire. (In the intense field between the poles of an electromagnet the wire in question was found to be distinctly paramagnetic, but so was a roll of paper; thus μ is taken as unity.)

These values put in (3) yield

$$\mathrm{R}'/\mathrm{R} = 31{\cdot}6. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Hence R for 1 centim. of the line, *i. e.* for 2 centim. of wire, is $2{\cdot}2 \times 10^5$; and we obtain

$$\mathrm{R}' \text{ per centim. of line} = 69{\cdot}5 \times 10^5. \quad . \quad . \qquad (5)$$

Now for the line in question $\mathrm{L}v$ has been determined *, both theoretically and experimentally, to be 56×10^{10} C.G.S. units.

Thus we have for the theoretical value of the attenuation constant

$$\sigma = \mathrm{R}'/2\mathrm{L}v = 0{\cdot}0000062, \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

which is only half that determined experimentally.

In the endeavour to account for this discrepancy two lines of thought are open to us.

(1) We may examine the order of accuracy of the experiments and inquire in which direction the determination probably errs; and (2) the applicability to the case in point of the theoretical expression given may be discussed, and the sense of the modification which it requires for rigorous application to the present case may perhaps be inferred. These will be taken in the above order.

* "Absorption of Electric Waves," Phil. Mag. Jan. 1897; Proc. Phys. Soc., Feb. and Mar. 1897.

The experimental value is calculated upon the supposition that the absorbing bridge used at the end of the line absorbs all the electric waves incident upon it, and that the reflecting bridge reflects all. If either supposition is incorrect the phenomena are changed, and the value found for the attenuation is in consequence affected.

Now it was previously found that the absorbing bridge for the line now in use must have a resistance of 560 ohms *. In the present work it was accordingly endeavoured to keep the resistance of the bridge at that value. It was adjusted before each occasion of its use and tested afterwards. The difference was usually of the order 2 or 3 ohms; but on one occasion it reached 10 ohms. However, to take an extreme case, let it be supposed that the critical value 560 is not the correct one, and that on some occasion of its use the bridge-resistance, though differing little from 560 ohms, differed by as much as 45 ohms from the value required for complete absorption of the electric waves incident upon it. Then the reflexion-coefficient of the bridge is given by †

$$\rho' = \frac{R_1 - Lv}{R_1 + Lv} = \frac{45}{2Lv + 45} = \frac{45}{1165}, \text{ nearly}, \quad . \quad . \quad . \quad (7)$$

where R_1 is the resistance of the bridge and Lv the instantaneous impedance of the line.

And it may easily be seen that the electrometer-throw when the absorbing bridge only is in use is, in consequence of its imperfect absorption, affected by a factor rather less than

$$(1+\rho'^2)/(1-\rho^2\rho'^2) = 1 + \tfrac{3}{2}\rho'^2, \text{ nearly}; \quad . \quad . \quad (8)$$

since ρ^2 is of the order one-half, ρ being the reflexion-coefficient of the oscillator.

Thus, for the supposed error of 45 ohms, which is an extreme case, we have $\rho' = 0{\cdot}04$ and the electrometer-throw increased in consequence by 0·24 per cent. But this is within the limits of the probable error of the determination of the ratios of the throws, and is therefore almost negligible.

Again, let us now examine the error consequent upon

* "Absorption of Electric Waves," Phil. Mag. Jan. 1897; Proc. Phys. Soc. Feb. and Mar. 1897.

† Heaviside's 'Electrical Papers,' vol. ii. pp. 132–133.

incomplete reflexion from the bridge used to reflect all. The resistance of this bridge should be zero. It consists of a copper wire put across the line; both it and the line were cleaned and polished at the places of contact, and the bridge made so as to spring on tightly into position. The bridge in question, put on with much less care than usual, showed a resistance of 0·04 ohm on testing with a P.O. box. But suppose by any mischance that the resistance of this bridge were half an ohm, then its reflexion-coefficient is given by

$$\rho'' = \frac{R_2 - Lv}{R_2 + Lv} = -\left(1 - \frac{2R_2}{Lv}\right) = -(1-\chi), \text{ say}, \quad . \quad (9)$$

where R_2 is the resistance of the bridge and Lv the instantaneous impedance of the line as before. Then the electrometer-throw when this bridge is in use is, in consequence of its imperfect reflexion, affected by a factor rather nearer unity than

$$\frac{1+\rho''^2}{2} \cdot \frac{1-\rho^2}{1-\rho^2\rho''^2} = \frac{1+(1-2\chi)}{2} \cdot \frac{1-\rho^2}{1-\rho^2(1-2\chi)} = 1 - 3\chi, \text{ nearly}. \quad (10)$$

And this factor, for half an ohm resistance, would diminish the electrometer-throw by 0·5 per cent. This again is almost negligible, being within the limits of the errors of determination of the ratios of the electrometer-throws.

Let us, however, in spite of the smallness of these effects on the electrometer-throw due to imperfect absorption or reflexion, inquire as to whether they are additive or compensatory. We at once see that they are additive, and that any departure from perfect absorption or perfect reflexion would result in a lessening of the observed ratio r of the electrometer-throws. Hence the true value of r which would be obtained under ideal conditions may be expected to exceed the actual one, if they differ at all.

In order to follow this possibility to its consequence, let us take each experimental value of r, *plus* its *probable error*, and from these data determine the corresponding value of σ. These probable errors, it will be noticed, exceed those which might be attributed to imperfection in the absorbing and reflecting arrangements, so entirely cover any errors which may be due to those sources. To still further test the consequences of extreme suppositions, values of the attenuation

have been calculated for a value of one ratio r_1 *plus* its probable error, and the other r_2 *minus* its probable error, and *vice versâ*. The results are scheduled in Table III.

Table III.

The effect on the Attenuation of possible errors in the Ratios of the Electrometer-throws.

Ratios of electrometer-throws with length of line after electrometer 48 metres, and length before 116 metres.	65 metres.	Value of s where attenuation factor $=10^{-s}$ per metre.	Value of σ where attenuation factor $=e^{-\sigma}$ per centim.	Remarks.
2·555	2·955	0·000566	0·0000130	Standard result.
2·599	2·980	0·000533	0·00001224	Result when each ratio is increased by its probable error.
2·599	2·930	0·000478	0·0000110	First ratio increased and second diminished by probable error.
2·511	2·980	0·000655	0·0000149	First ratio diminished and second increased by probable error.

It is thus seen that none of the suppositions considered bring the experimental value of the attenuation down to the theoretical value, though the slight difference made in the most probable case (second line of Table III.) is in the right direction, and lessens the value by nearly 6 per cent., whereas a reduction of 50 per cent. is required for agreement.

Turning now to the question of the validity of the expression for the effective resistance in the case under discussion: we have to notice (1) that this formula was developed for a wire whose return * was at a distance, and (2) that it is for "periodic currents following the harmonic law" †.

* Phil. Mag. May 1886, p. 390. † Ibid. p. 387.

Whereas in the actual experimental case we have

(1) The two wires comparatively near together (diameter 1·5 millim., distance of centres apart 8 centim.) ; and
(2) The wave-train is a rapidly damped one (see fig. 2).

To me it appears probable that each of these considerations if introduced into the theory would lead to a modification of the expression for the attenuation which would bring its value nearer the experimental one. Whether or not they would be competent to entirely bridge the discrepancy I do not pretend to say.

There are other possible causes of a discrepancy between the two values of σ. For example, any irregularity in the spacing or diameter of the wires would lead to a slight reflexion at the place of its occurrence, thus making the line behave towards the electric waves somewhat as a turbid or translucent medium behaves towards light. But as great care was taken in the spacing of the wires, no appreciable effect is likely to have arisen from this cause.

In conclusion I must thank Mr. Oliver Heaviside both for the suggestion which initiated this investigation and for his advice, ever freely given, during its continuance.

I am also specially indebted to one of our advanced students, Mr. Louis Lownds, for his able and painstaking assistance throughout the work both in the experimental and theoretical portions.

University College, Nottingham,
May 24, 1898.

XXV. *Diffusive Convection.*

By ALBERT GRIFFITHS, *M.Sc.* (*Vic.*), *A.R.C.S.* (*Lon.*). *

SECTION I.

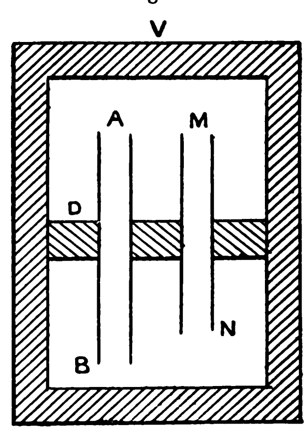
THE diffusion of a substance through a liquid, *e. g.* water, may be regarded as merely the transmission of the substance through the stationary water.

If the solution of the substance has a density different from that of water, depending on its strength, the act of diffusion tends to alter the density at any point: in general this tendency to alter the density prevents the possibility of statical equilibrium and hence convection-currents are produced.

So far as the author is aware the only case in which diffusion of the substance can take place without the production of convection-currents, is that in which diffusion takes place along a perfect cylinder the generating lines of which are truly vertical.

The determination of the magnitude of the convection-currents is, in general, a difficult task, involving, as it does, hydrodynamics as well as diffusion.

Fig. 1.



The present contribution determines the magnitude of the convection in a simple case, which arose in connexion with an attempt to devise an apparatus for the accurate determination of the coefficient of diffusion of copper sulphate through water.

Omitting some details, it may be said that the apparatus consists of a vessel, V, divided into two parts by a diaphragm D.

Through the diaphragm pass narrow tubes of which only two are shown in the figure. Means are taken for keeping the upper compartment full of nearly

* Read June 10, 1898.

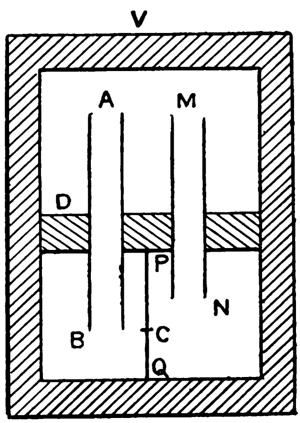
pure water, and the lower part full of a solution of copper sulphate.

Attention will be confined to two tubes AB, MN, only, and it will be assumed that the upper ends A and M are in contact with perfectly pure water, and that the lower ends B and N are in contact with a solution of copper sulphate of *constant* strength. Doubtless diffusion will tend to diminish the strength of the solution at B and N; but there is reason to believe that the diminution in strength, under the conditions of the experiment, is inappreciable.

In the diagram A and M are shown at the same level, but AB is longer than MN, and B is at a lower level than N.

Under these circumstances it will be shown that, when things have attained a steady state, there is a flow of liquid *up* the tube AB, and down the tube MN.

Fig. 2.



To simplify the mathematical treatment of the problem, it will be assumed that the coefficient of diffusion is a constant, and that the density, d, of the copper sulphate solution is given by the equation $d=1+t$, where t is the quantity of copper sulphate per cubic centimetre of solution.

To show in a simple manner that the flow indicated must take place, imagine that the lower compartment is divided into two parts by a vertical partition, PQ, and that at C on a level with B there is placed a tap which can be opened or closed at will.

Let L_1=length of AB,
" L_2= " MN,
" l =distance of a point in AB from the top A,
" T =quantity of copper sulphate ($CuSO_4$) per cub. centim. in the lower compartment of the vessel V,
" g =value of gravity.

When the tap at C is closed and the diffusion has attained

the steady state, the density is given by the equation

$$d=1+t,$$

whence in the case of the tube AB,

$$d=1+\frac{l}{L_1}T.$$

The pressure at C on the *left*-hand side of the partition, neglecting the initial pressure at the upper ends of the tubes, is given by the equation

$$P_1=g\int_0^{L_1}\left(1+\frac{l}{L_1}T\right)dl$$
$$=gL_1\left(1+\frac{T}{2}\right).$$

The pressure on the *right*-hand side of C is given by the equation

$$P_2=gL_2\left(1+\frac{T}{2}\right)+g(L_1-L_2)(1+T).$$

P_2 is greater than P_1, the difference being equal to

$$\frac{gT(L_1-L_2)}{2}.$$

Since P_2 exceeds P_1, if the tap be opened liquid will flow from the right to the left.

If the tap be now closed, the diffusion will again attain a steady state, and the same inequality of pressure will be developed as before. The tap may again be opened and so on.

It is not necessary, however, to have a tap at C; if there is no obstacle whatever between B and N a steady flow will take place down MN and up AB.

The inequality between AB and MN produces, as we have seen, a pressure tending to produce circulation. On the assumption that the density is the same at all points of a horizontal layer, the magnitude of this pressure in dynes per square centim. equals

$$g\int_0^{L_2} d\,\partial l+g(L_1-L_2)(1+T)-g\int_0^{L_1} d\,\partial l,$$

where d is a function of the point under consideration in the

tube AB or MN. To find the nature of this function the diffusion of copper sulphate, in a vertical tube, along which the liquid is moving, must be studied.

Section II.

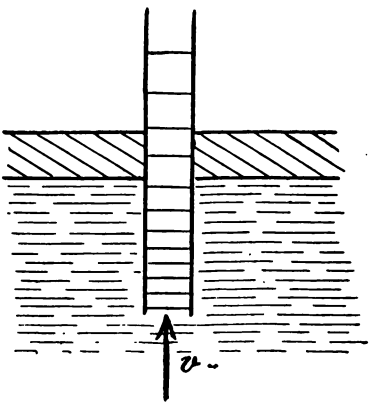
Diffusion of a salt in a vertical tube through which the liquid is flowing with a constant velocity, when the steady state has been attained.

Let l = distance of point from top of tube.
„ L = total length of tube.
„ A = sectional area of tube.
„ k = coefficient of diffusion.
„ t = quantity of dissolved salt per cub. centim. at a distance l from the top.
„ v = velocity of liquid up the tube.

When the steady state has been attained, the quantity of salt which passes in a second is a constant, and equals

$$vAt + Ak\frac{\partial t}{\partial l}.$$

Fig. 3.



If t is zero at the top of the tube, and T at the bottom of the tube, the solution of the differential equation which arises is

$$l = \frac{k}{v}\log_e\frac{T}{T - t\left(1 - e^{-\frac{vL}{k}}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$t=\mathrm{T}\frac{1-e^{-\frac{vl}{k}}}{1-e^{-\frac{v\mathrm{L}}{k}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By differentiating (1) and (2)

$$\frac{\partial t}{\partial l}=\frac{v}{k}\,\frac{\mathrm{T}-t\left(1-e^{-\frac{vl}{k}}\right)}{1-e^{-\frac{v\mathrm{L}}{k}}},$$

and substituting this value of $\frac{\partial t}{\partial l}$ in the expression $v\mathrm{A}t+\mathrm{A}k\frac{\partial t}{\partial l}$, we find for the *quantity* of substance which passes per second

$$\frac{v\mathrm{AT}}{1-e^{-\frac{v\mathrm{L}}{k}}}.$$

Let p=difference of pressure between bottom and top of tube, neglecting viscosity,

$$\begin{aligned}p&=g\int_0^{\mathrm{L}} d\,\partial l\\ &=g\int_0^{\mathrm{L}}(1+t)\partial l\\ &=g\int_0^{\mathrm{L}}\left\{1+\mathrm{T}\frac{1-e^{-\frac{vl}{k}}}{1-e^{-\frac{v\mathrm{L}}{k}}}\right\}\partial l, \text{ from (2).}\end{aligned}$$

Integrating

$$p=g\mathrm{L}+\frac{g\mathrm{TL}}{1-e^{-\frac{v\mathrm{L}}{k}}}-\frac{kg\mathrm{T}}{v}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When the velocity is *downwards*

$$p=g\mathrm{L}+\frac{g\mathrm{TL}}{1-e^{\frac{v\mathrm{L}}{k}}}+\frac{kg\mathrm{T}}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Section III.

Determination of the magnitude of the convection-current in the case of tubes of equal bore, neglecting viscosity.

From (3) and (4), and from the expression in Section I., it can be seen that the pressure tending to produce circulation

is given by the expression

$$P = gL_2 + \frac{gTL_2}{1-e^{\frac{vL_2}{k}}} + \frac{kgT}{v} + g(1+T)(L_1 - L_2)$$

$$-gL_1 - \frac{gTL_1}{1-e^{-\frac{vL_1}{k}}} + \frac{kgT}{v},$$

or

$$P = gT(L_1 - L_2) + \frac{2kgT}{v} + \frac{gTL_2}{1-e^{\frac{vL_2}{k}}} - \frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}. \quad . \quad (5)$$

This pressure never can become negative in the positive direction; the limiting value of v is therefore given by the equation

$$gT(L_1 - L_2) + \frac{2kgT}{v} + \frac{gTL_2}{1-e^{\frac{vL_2}{k}}} - \frac{gTL_1}{1-e^{-\frac{vL_1}{k}}} = 0. \quad . \quad . \quad (6)$$

In general the solution of this equation will involve considerable labour, but if $\frac{vL_1}{k}$ and $\frac{vL_2}{k}$ are small, it can be shown that

$$v = \frac{6k(L_1 - L_2)}{L_1^2 + L_2^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or, if $L_1 - L_2 = \delta L$, and either length $= L$ approximately,

$$v = \frac{3k\delta L}{L^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As a practical example take

$k = 2{\cdot}47 \times 10^{-6}$, the value of the coefficient of diffusion for copper sulphate,
$L = 4$ centim.,
$\delta L = 0{\cdot}05$ centim., *i.e.* half a millimetre.

Substituting in (8) we obtain

$v = 2{\cdot}306 \times 10^{-8}$ centim. per second
$= 5{\cdot}09$ centim. per year.

The author has tested the accuracy of this approximation in the original equation (6), and with seven figure logarithms has found the value of v to be exactly correct. He finds that (8) gives a value correct to a fraction of a per cent.

SECTION IV.

Effect of Diffusive Convection on the quantity of dissolved substance transmitted when the tubes are of equal bore.

The quantity transmitted, per second, of the tube L_1 equals from Section II.

$$\frac{vAT}{1-e^{-\frac{vL_1}{k}}};$$

expanding by the exponential theorem and dividing out we obtain

$$\frac{kAT}{L_1}\left(1+\frac{1}{2}\frac{vL_1}{k}+\frac{1}{12}\frac{v^2L_1^2}{k^2}+\&c.\right).$$

It is obvious that the expression within brackets is the correcting factor for the velocity of the liquid up the tube.

Taking the case considered in the previous section, with $L_1=4$ centim., and $v=2{\cdot}306\times10^{-8}$ centim. per second, the value of this correcting factor works out as 1·019 approximately; which shows that a velocity of about 5 centim. per year up a tube 4 centim. long produces an increase in the quantity of copper sulphate transmitted of about 2 per cent. This is a forcible illustration of the necessity for complete rest in diffusion experiments.

What is of chief importance here, however, is not the error produced in one tube, but the resultant error due to the flow in the two tubes.

The quantity transmitted along the two tubes is

$$\frac{vAT}{1-e^{-\frac{vL_1}{k}}}-\frac{vAT}{1-e^{\frac{vL_2}{k}}}$$

$$=\frac{kAT}{L_1}\left(1+\frac{1}{2}\frac{vL_1}{k}+\frac{1}{12}\frac{v^2L_1^2}{k^2}\right)+\frac{kAT}{L_2}\left(1-\frac{1}{2}\frac{vL_2}{k}+\frac{1}{12}\frac{v^2L_2^2}{k^2}\right)$$

$$=\frac{kAT}{L_1}\left(1+\frac{1}{12}\frac{v^2L_1^2}{k^2}\right)+\frac{kAT}{L_2}\left(1+\frac{1}{12}\frac{v^2L_2^2}{k^2}\right).$$

The correcting factor equals $1+\frac{1}{12}\frac{v^2L_1^2}{k^2}$ approximately; or, substituting the value obtained for v, $1+\frac{3}{4}\left(\frac{\delta L}{L_1}\right)^2$. Sub-

stituting the values given above for δL and L_1, the value obtained for the correcting factor is 1·00012 approximately; or the error due to the convection-current equals about one-hundredth of a per cent.

Section V.

Tubes of unequal bore.

Let A = sectional area of the longer tube of length L_1 and
r^2A = sectional area of the shorter tube of length L_2.
Let v = velocity of flow in the shorter tube, then
r^2v = velocity of flow in the longer tube.

By algebraical considerations similar to those already employed it can be shown that

$$v=\frac{6k\delta L}{L_2^2+r^2L_1^2}$$

$$r^2v=\frac{6kr^2\delta L}{L_2^2+r^2L_1^2}.$$

When $r^2=0$, the velocity in the shorter tube equals $\frac{6k\delta L}{L_2^2}$, whilst the velocity in the longer tube is zero.

When $r^2=\infty$, the velocity in the shorter tube is zero, and the velocity in the longer tube is $\frac{6k\delta L}{L_1^2}$.

The maximum value of the velocity along a tube occurs when its bore is infinitely small in comparison with that of the other tube. In both cases the maximum value (since δL is small) equals $\frac{6k\delta L}{L_1^2}$ approximately; *i. e.* the maximum value is double the velocity produced when the tubes are of equal bore.

It can be shown by algebraical transformations that the correcting factor equals

$$1+\frac{3(\delta L)^2\times L_1L_2}{\left(\frac{L_2^2}{r}+rL_1^2\right)}.$$

It is unity when r equals infinity or zero. Its *maximum* value occurs when $r=\frac{L_2}{L_1}$ (*i.e.* when the diameters of the

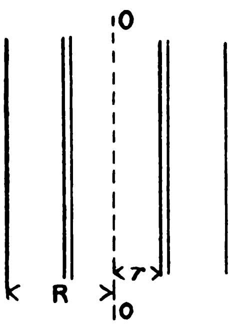
tubes are in the same proportion as their lengths) and is $1+\frac{3}{4}\frac{(\delta L)^2}{L_1^2}$ approximately; it is seen that the maximum value is practically identical with the value in the case of tubes of equal bore.

Section VI.

Probable effect of Viscosity on the Convective Flow.

The internal friction of the moving liquid will diminish, to some extent, the velocity. It is obvious that the liquid in the interior portion of a tube will travel at a faster rate than that near the walls, and there is little doubt that the surfaces of equal density will not be horizontal planes. In all probability therefore the diffusion will take place radially as well as axially. It is also possible that the stream-lines will no longer be vertical; but superimposed on the vertical motion there may be slight eddies.

Fig. 4.



The author has not attempted the general problem, and in what follows only the ideal case is considered, in which both diffusion and motion take place in a vertical direction, *i. e.* their radial components are neglected.

Let OO′ represent the axis of the tube of radius R.

By considering the space between two concentric cylinders of radii r and δr, we obtain the equation

$$-\frac{\partial^2 v}{\partial r^2}\delta r\times 2\pi r L\eta+\left(gL+\frac{gTL}{1-e^{-\frac{vL}{k}}}-\frac{kgT}{v}\right)\times 2\pi r\delta r$$

$$=p\times 2\pi r\delta r, \quad . \quad . \quad . \quad (9)$$

where v = the upward velocity,

η = the coefficient of viscosity,

p = difference between the pressures at the lower and upper ends of the tube.

Let $p=\left(1+\frac{T}{2}\right)gL+P$, *i. e.* P is the excess of pressure above that necessary to produce statical equilibrium.

By algebraical transformation, neglecting the square and higher powers of $\frac{v}{k}$, equation (9) becomes

$$\frac{\partial^2 v}{\partial r^2}-\frac{1}{12}\frac{gTL}{\eta k}v=\frac{-P}{L\eta}. \quad . \quad . \quad . \quad . \quad (10)$$

Let

$$\frac{1}{12}\frac{gTL}{\eta k}=f^2.$$

The solution of (10) is

$$v=M_1e^{fr}+M_2e^{-fr}+\frac{12Pk}{gTL^2}. \quad . \quad . \quad . \quad (11)$$

If there is no slipping at the walls of the tube $v=0$, when $r=R$, hence $0=M_1e^{fR}+M_2e^{-fR}+\frac{12Pk}{gTL^2}$; when

$$r=0,\ \frac{\partial v}{\partial r}=0,\ \text{hence}$$

$$0=M_1-M_2.$$

Ultimately (11) becomes

$$v=\frac{12Pk}{gTL^2}\left(1-\frac{e^{fr}+e^{-fr}}{e^{fR}+e^{-fR}}\right).$$

If $\eta=0$, it can be shown algebraically that

$$v=\frac{12Pk}{gTL^2}.$$

What may be called the proportional-deviation equals

$$\frac{e^{fr}+e^{-fr}}{e^{fR}+e^{-fR}}.$$

To take a definite example let

$$\begin{aligned} R&=0{\cdot}1 \text{ centim.},\\ \eta&=0{\cdot}1324,\\ T&=0{\cdot}1,\\ k&=2{\cdot}47\times10^{-6},\\ L&=4 \text{ centim.}\end{aligned}$$

Then $f=10{,}000$, and a study of the deviation shows that it is practically *nil*, except when r is nearly equal to R; in which case it equals

$$e^{f(r-\mathrm{R})} \text{ approximately.}$$

To make it equal to 0·001, r must equal 0·09931, and it is clear that, on the assumptions made, the effect of the viscosity is to produce variations in the average flow of only a fraction of a per cent. With tubes of ordinary calibre, no appreciable error is made if the viscosity be neglected altogether. The expressions also show that the tubes would have to be very narrow to affect the magnitude of the diffusive convection to any appreciable extent.

Another method of showing that very little error is made by neglecting the viscosity is to obtain an expression for the average velocity.

Let $\mathrm{V}=$ the average velocity up the tube, then

$$\mathrm{V}\times\pi\mathrm{R}^2=\int_0^{\mathrm{R}} 2\pi r v\,\partial r$$

$$=\frac{12\mathrm{P}k}{g\mathrm{TL}^2}\left[\frac{\pi\mathrm{R}^2-2\pi\left\{\frac{\mathrm{R}}{f}(e^{f\mathrm{R}}-e^{-f\mathrm{R}})-\frac{1}{f^2}(e^{f\mathrm{R}}+e^{-f\mathrm{R}})+\frac{2}{f^2}\right\}}{e^{f\mathrm{R}}+e^{-f\mathrm{R}}}\right].$$

Since f is large the second term within the square brackets may be neglected, and we obtain

$$\mathrm{V}=\frac{12\mathrm{P}k}{g\mathrm{TL}^2} \text{ approximately.}$$

Considering the case of two tubes of equal bore, but of unequal length, studied in Section III.,

let $\mathrm{P}_1=$ "excess of pressure" producing the flow up the tube L_1.

$\mathrm{P}_2=$ the corresponding excess (or defect) producing the flow down the tube L_2,

then it can be readily seen that

$$\mathrm{P}_2=\frac{g\mathrm{T}}{2}(\mathrm{L}_1-\mathrm{L}_2)-\mathrm{P}_1\,;$$

hence
$$\frac{g\mathrm{TL}_2{}^2\mathrm{V}_2}{12k}=\frac{g\mathrm{T}(\mathrm{L}_1-\mathrm{L}_2)}{2}-\frac{g\mathrm{TL}_1{}^2\mathrm{V}}{12k}.$$

Putting $V_1 = V_2$, we obtain

$$V_1 = \frac{6k(L_1 - L_2)}{L_1^2 + L_2^2}.$$

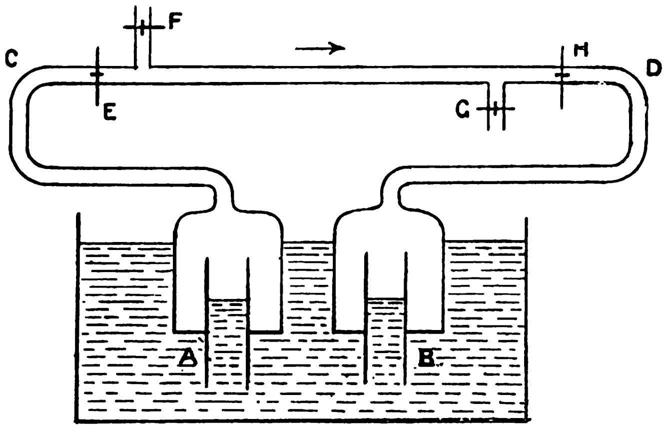
This is the same result as that obtained in Section III., when viscosity was neglected.

Section VII.

An Experimental Illustration of Diffusive Velocity.

The adjoining figure is a diagrammatic sketch of an apparatus kindly made by Mr. F. W. Rixon to test the existence of diffusive convection.

Fig. 5.



The upper extremities of two tubes, A and B, of equal diameter but slightly unequal in length, terminate in compartments connected by a capillary tube CD.

Any motion down B and up A is magnified in the proportion of the sectional area of A or B to the sectional area of the capillary tube.

The motion along CD is detected by the addition of copying-ink or other colouring matter. To introduce the ink, the taps E and H are closed, G and F opened, and the ink poured down F.

Owing to viscosity the velocity of the liquid at the centre is *double* the average velocity.

In the experiments, the results of which are recorded below, the tubes A, B, &c. were filled with water and placed in a strong solution of copper sulphate, the taps E, F, G, and H being closed. The taps E and H were then opened, and a rush would doubtless occur along CD. After some time, perhaps 5 or 10 minutes, the taps E and H were closed, and the ink added.

The chief data are as follows :—

Diameter of A or B = 1·09 centim. approximately.
Diameter of capillary = 0·019 centim. approximately.
Height of top of B above top of A = 0·08 centim.
Height of bottom of B above bottom of A = 0·292 centim.
Length of A = 4·622 centim.
Length of B = 4·410 centim.

The motion of the centre of the liquid in the capillary is indicated in the table below.

Hours from start ...	1	18½	43	66	90	114	162	189½	211	263	283	354.
Motion of centre in centim.	4·9	8·6	14·2	18·4	22·2	25·5	32·3	35·9	39·0	45·6	48·8	56·35.

If a curve be drawn with hours as abscissæ, and the distance moved as ordinates, a regular diminution of the flow is indicated, the velocity tending to a steady value. The average velocity in the last three days was 2·5 centim. per day.

Calculations made on the principles indicated in the previous section give about ·8 centim. per day as the velocity.

The observed velocity is thus three times the calculated. As the apparatus when originally put together was only intended to give qualitative results, it would be premature to discuss the cause of the difference *.

* *Note added July 15th*, 1898.—Later experiments indicate that the difference between the calculated and observed velocities in the above was due to the continual increase of density of the solution owing to evaporation of water at the surface. On covering the surface with oil, to prevent evaporation, the observed and calculated values are found to be close enough to *prove the substantial truth of the theory*. Thus in one experiment, the observed motion of the index, at the end of a fortnight, was 2·3 centim. per day, whereas theory indicates that the final velocity should be 2·1 centim. per day.

After the above observations had been made, the tube B was lowered with respect to A; the flow along the capillary from left to right was stopped, and there was probably a flow in the opposite direction. (The ink is not adapted for detecting slight backward motions.)

A second apparatus made by Mr. Rixon was fitted up, and a solution of common salt employed. Currents were produced in the direction indicated by theory. This new apparatus is capable of delicate adjustment, and experiments with it are still in progress. To the author the interest lies not so much in the confirmation of the theory as in the possibility of a physical method of determining the coefficient of diffusion.

A theoretical consideration suggested by the phenomena of diffusive convection may not be uninteresting.

As the water flows along the capillary, friction occurs and heat is produced. Ultimately the dissolved salt mingles completely with the water. Instead of allowing the flow to occur along the capillary the tap E or H might be closed. The dissolved salt would then mingle completely with the water without producing any flow, and the final state (assuming the apparatus is thermally insulated) is exactly the same as before.

Now the principle of the conservation of energy indicates that the heat produced in the capillary plus the heat developed by the mixing of the solution with water is a constant.

Does not this indicate that the heat produced on mixing a solution with water depends on how the mixing takes place? Is the matter connected with a sort of surface-tension existing in the spaces between a strong and weak solution?

XXVI. *Theory of the Hall Effect in a Binary Electrolyte.* *By* F. G. DONNAN, *M.A., Ph.D.**

IN 1883 Roiti † investigated the subject of a possible Hall effect in electrolytic solutions, but failed to obtain any positive result. Recently, however, the question has been taken up by Bagard ‡, who has obtained very considerable effects in aqueous solutions of zinc and copper sulphates. On the other hand, negative results have been obtained by Florio §, and as both Bagard and Florio maintain the correctness of their experimental work, a polemic on the subject has arisen between them. In this condition of affairs it seemed worth while to examine what effect might be expected theoretically. With this purpose in view I made the calculation contained in the following pages. Subsequently I discovered that a similar theory had been given by Van Everdingen, jun.‖; but as I do not arrive at quite the same results, and have considered the subject somewhat more generally, it seemed to me not to be needless repetition to communicate this note.

The basis of the following calculation may be stated briefly as follows. The diagram shows the directions of the primary current and the magnetic field. The lines of flow of the primary current are supposed to be straight and the magnetic field everywhere uniform. The effect of the ponderomotive forces on the moving ions in the magnetic field is to urge both positive and negative ions in the positive direction of z. The moving ions will thus acquire component velocities in this direction, and these velocities being in general different for each sort of ion, the result is the separation of positive and negative charges, whereby a potential-gradient is set up in the direction of the z-axis which reduces the originally unequal velocities to equality. We thus obtain a flow of ionic matter in the positive z-direction. This produces in

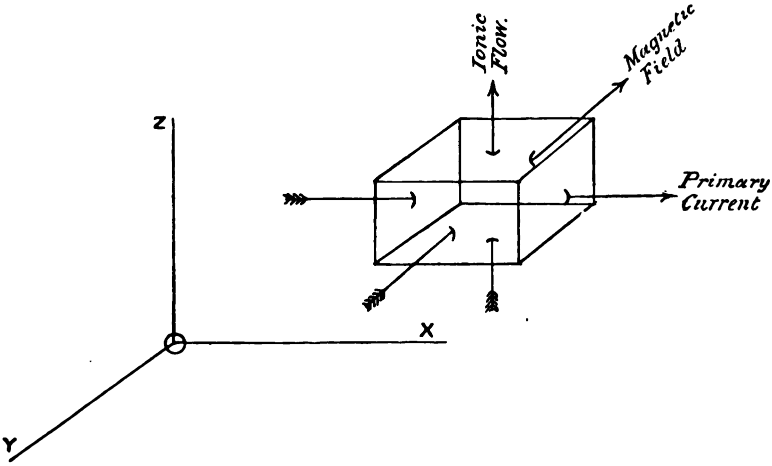
* Read June 24, 1898. † *Journ. de Physique*, 1883.

‡ *Comptes Rendus*, vol. cxxii. pp. 77–79, and cxxiii. pp. 1270–1273 (1896).

§ *Nuovo Cimento*, [4] vol. iv. pp. 106–111 (1896).

‖ *Metingen over het Verschijnsel van Hall en de Toename van den Weerstand in het Magnetisch Veld*, p. 102 *et seq.* (Leiden, 1897).

its turn a reverse osmotic gradient or concentration-fall both for dissociated and undissociated salt. The stationary state is finally attained when the net flux of ionic matter in the positive direction of z is balanced by the flux of undissociated salt in the opposite direction.



Let J = current-density of primary current,
c = concentration (in mols.) of positive *plus* negative ions,
p = corresponding osmotic pressure,
C = concentration (in mols.) of undissociated salt,
P = corresponding osmotic pressure,
u, v = velocities in centim. per second acquired under unit force by one gram-mol. of positive and negative ionic matter respectively,
G = velocity acquired under unit force by one gram-mol. of undissociated salt,
$\frac{d\pi}{dx}$ = potential-gradient of primary current,
$\frac{de}{dz}$ = potential-gradient of Hall-effect,
ϵ = quantity of electricity travelling per gram-equivalent of ionic matter,
ω = valency of each ion,
H = magnetic field-strength,
t = temperature (absolute).

We shall suppose the laws $p=c\mathrm{R}t$ and $\mathrm{P}=\mathrm{CR}t$ to hold for the solution in question, although this is not essential. We have furthermore

$$\mathrm{J}=-\frac{\omega^2\epsilon^2 c}{2}(u+v)\frac{d\pi}{dx}. \quad . \quad . \quad . \quad . \quad . \quad (\text{i.})$$

Now an ion with a positive charge ωe moving with a velocity V in the positive direction of axis of x will be acted on by a force in the positive z direction equal to $\omega\epsilon\mathrm{VH}$; so that the force in this direction on a positive gram-ion is given by $-\omega^2\epsilon^2 u\mathrm{H}\frac{d\pi}{dx}$, whence it has a component velocity in the same direction amounting to $-\omega^2\epsilon^2 u^2\mathrm{H}\frac{d\pi}{dx}$. By similar reasoning the velocity of a negative gram-ion in the positive z direction is $-\omega^2\epsilon^2 v^2\mathrm{H}\frac{d\pi}{dx}$.

Expressing quantities of matter in mols., we can now draw up the following list, which takes into account all the fluxes of matter occurring in the solution.

1. Quantity of positive ionic matter traversing per unit time unit section perpendicular to axis of z in positive direction of this axis, due to ponderomotive forces arising from the magnetic field.	$=-\frac{\omega^2\epsilon^2 c}{2}u^2\mathrm{H}\frac{d\pi}{dx}$.
2. Corresponding quantity of negative ionic matter.	$=-\frac{\omega^2\epsilon^2 c}{2}v^2\mathrm{H}\frac{d\pi}{dx}$.
3. Flux of positive ionic matter in same direction due to potential-gradient along z-axis.	$=-\frac{\omega\epsilon c}{2}u\frac{de}{dz}$.
4. Corresponding flux for negative ionic matter.	$=+\frac{\omega\epsilon c}{2}v\frac{de}{dz}$.
5. Flux of positive ionic matter in same direction due to osmotic gradient.	$=-u\frac{\mathrm{R}t}{2}\frac{dc}{dz}$.
6. Corresponding flux of negative ionic matter.	$=-v\frac{\mathrm{R}t}{2}\frac{dc}{dz}$.
7. Flux of undissociated salt in positive z direction due to osmotic gradient.	$=-\mathrm{GR}t\frac{d\mathrm{C}}{dz}$.

The stationary condition then gives the equations:—

$$\frac{\omega^2\epsilon^2 c}{2}u^2\mathrm{H}\frac{d\pi}{dx}+\frac{\omega\epsilon c}{2}u\frac{de}{dz}+\frac{u\mathrm{R}t}{2}\frac{dc}{dz}+\mathrm{GR}t\frac{d\mathrm{C}}{dz}=0, \quad . \quad \text{(ii.)}$$

$$\frac{\omega^2\epsilon^2 c}{2}v^2\mathrm{H}\frac{d\pi}{dx}-\frac{\omega\epsilon c}{2}v\frac{de}{dz}+\frac{v\mathrm{R}t}{2}\frac{dc}{dz}+\mathrm{GR}t\frac{d\mathrm{C}}{dz}=0. \quad . \quad \text{(iii.)}$$

Elimination of $\frac{d\pi}{dx}$ and $\frac{d\mathrm{C}}{dz}$ from the equations (i.), (ii.), and (iii.) leads to the result

$$\frac{de}{dz}=\frac{2}{\omega\epsilon c}\frac{u-v}{u+v}\mathrm{HJ}-\frac{\mathrm{R}t}{\omega\epsilon c}\frac{u-v}{u+v}\cdot\frac{dc}{dz}. \quad . \quad . \quad . \quad \text{(iv.)}$$

It is to be observed that equation (iv.) holds for the variable as well as the stationary state, because although during the variable state the left-hand members of equations (ii.) and (iii.) are not zero, they are always equal to each other.

Accordingly for the initial phenomenon, before any appreciable concentration-gradient occurs, we obtain, putting $c=c_0$ and $\frac{dc}{dz}=0$,

$$\frac{de}{dz}=\frac{2}{\omega\epsilon c_0}\frac{u-v}{u+v}\mathrm{HJ}=\omega\epsilon(v-u)\mathrm{H}\frac{d\pi}{dx}, \quad . \quad . \quad \text{(v.)}$$

or, integrating,

$$\iint\frac{de}{dz}dy\,dz=\frac{2\mathrm{H}}{\omega\epsilon c_0}\frac{u-v}{u+v}\iint \mathrm{J}dy\,dz\,;$$

whence

$$e=\frac{2}{\omega\epsilon c_0}\frac{u-v}{u+v}\frac{\mathrm{H}i}{d}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi.)}$$

where e = total difference of potential measured in direction of positive z-axis, d = thickness measured in direction of magnetic field, and i = primary current-strength. So that we obtain for the constant of the initial Hall-effect the value

$$\text{const.}=\frac{2}{\omega\epsilon c_0}\frac{u-v}{u+v}.$$

In order to further investigate the stationary state, we shall suppose that the equilibrium-equation required by the laws of electrolytic dissociation is everywhere satisfied. This amounts

to supposing that the processes which adjust this equilibrium proceed very much more rapidly than any of the other changes occurring in the system. In the uncertain state of knowledge concerning the equilibrium-equation it may be written for the present $C=\phi(c)$; so that $\frac{dC}{dz}=\phi'(c)\frac{dc}{dz}$. Eliminating $\frac{dC}{dz}$ between this equation and (ii.) and between this equation and (iii.), we obtain the equations

$$\frac{de}{dz}=\frac{2}{\omega\epsilon c}\frac{u}{u+v}\mathrm{HJ}-\frac{\mathrm{R}t}{\omega\epsilon c}\left\{\frac{2\mathrm{G}\phi'(c)}{u}+1\right\}\frac{dc}{dz}, \quad \text{(vii.)}$$

$$\frac{de}{dz}=-\frac{2}{\omega\epsilon c}\frac{v}{u+v}\mathrm{HJ}+\frac{\mathrm{R}t}{\omega\epsilon c}\left\{\frac{2\mathrm{G}\phi'(c)}{v}+1\right\}\frac{dc}{dz}. \quad \text{(viii.)}$$

Writing $\frac{2\mathrm{G}\phi'(c)}{u}+1=\mathrm{L}$, $\frac{2\mathrm{G}\phi'(c)}{v}+1=\mathrm{M}$, we obtain from (vii.) and (viii.)

$$\frac{de}{dz}=\frac{2}{\omega\epsilon c}\frac{1}{\mathrm{L}+\mathrm{M}}\frac{\mathrm{M}u-\mathrm{L}v}{u+v}\mathrm{HJ},$$

or

$$\left.\begin{aligned}\frac{de}{dz}&=\frac{2}{\omega\epsilon c}\frac{1}{\mathrm{L}+\mathrm{M}}\frac{\mathrm{M}u-\mathrm{L}v}{u+v}\mathrm{HJ},\\ \frac{de}{dz}&=\frac{\omega\epsilon}{\mathrm{L}+\mathrm{M}}(\mathrm{L}v-\mathrm{M}u)\mathrm{H}\frac{d\pi}{dx}.\end{aligned}\right\} \quad \text{(ix.)}$$

For a completely dissociated electrolyte $\mathrm{L}=1$, $\mathrm{M}=1$, and we get instead of equations (ix.)

$$\frac{de}{dz}=\frac{1}{\omega\epsilon c}\frac{u-v}{u+v}\mathrm{HJ}=\frac{\omega\epsilon}{2}(v-u)\mathrm{H}\frac{d\pi}{dx}. \quad \text{(ix. }a\text{)}$$

Comparing equations (v.) and (ix. a), we see that the final potential-gradient for a completely dissociated electrolyte is just one-half the initial gradient.

Let $\frac{de}{dz}\Big/\frac{d\pi}{dx}=\mathrm{D}$, and denote by U, V the velocities acquired under unit potential gradient by a gram-mol. of positive and negative ionic matter respectively. Equation (ix. a) may then be written

$$\mathrm{D}=\tfrac{1}{2}(\mathrm{V}-\mathrm{U})\mathrm{H}. \quad \text{(x.)}$$

Van Everdingen * arrived at the equation $\mathrm{D}=(\mathrm{V}-\mathrm{U})\mathrm{H}$

* *Loc. cit.* p. 207.

for the stationary state in an electrolyte supposed to be very slightly dissociated; but he assumes in this case that the ionic concentration does not vary throughout the solution, an assumption which is inconsistent with the equations (vii.) and (viii.), and therefore appears to me to be erroneous.

Equation (x.) may now be applied to the data obtained by Bagard. He finds, for example, the following results.

$CuSO_4$ solution. Concentration $=\frac{1}{16}$ grm. equiv. per litre. Temperature 21°–26° C.

H.	D.	$\frac{D}{H}$.
385 C.G.S. units.	·0018	$\cdot 46 \times 10^{-5}$
707 " "	·0024	$\cdot 34 \times 10^{-5}$
962 " "	·0034	$\cdot 35 \times 10^{-5}$

To test these results by means of the theory, it is only necessary to calculate the value of $\frac{1}{2}(V-U)$, where it must be noticed that V and U are the velocities of the gram-mols. in centim. per second under a potential-gradient of one C.G.S. electromagnetic unit of potential per centim. From Ostwald's *Lehrbuch*, vol. ii. p. 770, the molecular conductivity of $CuSO_4$ solutions for the highest dilution at 18° C. is 217 in Siemens units, and therefore 230 in mho's. As the calculation is only very approximate, we may put consequently

$$(U+V)_{18^\circ}=\frac{230}{96540\times 2\times 10^8}.$$

From the same source, p. 612, we get $\frac{U}{U+V}=\cdot 36$, a value which may be regarded as fairly correct for temperatures in the neighbourhood of 20° C. (according to Hittorf and Bein, *loc. cit.*). Hence we obtain finally

$$\tfrac{1}{2}(V-U)_{18^\circ}=16\times 10^{-13}.$$

The value of $\frac{1}{2}(V-U)$ for temperatures 21°–26° C. will probably be somewhat smaller. Of course the solution of concentration $=\frac{1}{16}$ grm.-equiv. per litre is not by any means

completely dissociated; and by using the proper equation, namely,

$$\frac{D}{H} = \omega\epsilon \frac{Lv - Mu}{L + M},$$

since $v > u$ and therefore $\frac{Lv - Mu}{L + M} > \frac{v - u}{2}$, we should obtain a higher value for $\frac{D}{H}$. Nevertheless this would never account for the difference between $\cdot 39 \times 10^{-5}$ as observed by Bagard and 16×10^{-13} as deduced theoretically for the case of complete dissociation.

So far as I can see, the theory here given is wholly in favour of the negative results obtained by Roiti and Florio; and it would therefore seem that Bagard has measured a phenomenon not contemplated in the foregoing theory. Van Everdingen in the paper referred to above supports Bagard; but this is owing to his having accidentally omitted the factor 10^{-8} in his numerical work.

The ionic concentration-fall can be readily calculated for a completely dissociated electrolyte. From equations (iv.) and (ix. a) we get at once

$$Rt\frac{dc}{dz} = HJ,$$

or

$$\frac{d \log c}{dz} = -\frac{\omega^2\epsilon^2}{2Rt}(u + v)H\frac{d\pi}{dx};$$

whence

$$\frac{1}{z_2 - z_1}\log\frac{c_1}{c_2} = \frac{\omega\epsilon(U + V)}{2Rt}H\frac{\pi_2 - \pi_1}{x_2 - x_1};$$

or

$$\frac{2Rt}{\omega\epsilon}\log\frac{c_1}{c_2} = H(U + V)(z_2 - z_1)\frac{\pi_2 - \pi_1}{x_2 - x_1}.$$

Thus, calling E the P.D. between two transverse electrodes (of the same metal as the kation) due only to the differences in concentration set up (a case which could be realized by taking an electrolyte such as silver nitrate, for which U and V are very nearly equal), it follows that

$$E = \tfrac{1}{2}H(U + V)(z_2 - z_1)\frac{\pi_2 - \pi_1}{x_2 - x_1}.$$

Taking the case of copper sulphate, we get, as before,

$$U+V=\cdot 00119.$$

Putting $z_2-z_1=1$ centim., and $\frac{\pi_2-\pi_1}{x_2-x_1}=1$ (volt per centim.), and $H=20,000$ C.G.S. units, we obtain

$$E=12\times 10^{-8} \text{ volts}.$$

Hence it would appear that in this, or in any other similar experimental arrangement, it would be necessary not only to employ an extremely strong magnetic field, but also a very high primary potential-gradient, *i. e.* of the order of 10,000 volts per centim. This might perhaps be realized experimentally by employing very dilute solutions and a large accumulator-battery, such as that recently employed by Professors Trowbridge and Richards *.

20th May, 1898.

Note added May 27th.—Dr. Van Everdingen informs me that he has discovered the slip in his calculation, and agrees with the remarks made above concerning the experiments of Florio and Bagard.

XXVII. *Apparatus illustrating the Action of Two Coupled Electric Motors. By* Prof. Carus Wilson.†

[Abstract.]

THIS is an apparatus to illustrate the action of two electric motors coupled in such a way as to admit of their rotating at different speeds. The two shafts are placed in line, and each fitted with a bevel wheel gearing into an intermediate wheel, the axis of which is at right angles to the line of the motor shafts, and free to rotate in a plane at right angles to that line.

The motors can be made to rotate at different speeds by altering the strength of the magnets of either or both. The motion of the intermediate wheel depends upon the difference of the two speeds if the two motors are rotating in opposite

* Phil. Mag. February 1897.

† Read June 24, 1898.

directions, and upon their mean if they are rotating in the same direction.

A simple graphic construction enables us to predetermine the action of the arrangement with any given load on the intermediate wheel. Calling the two motors A and B, and the intermediate wheel C, we can draw lines representing, on a base of current, the speeds and the torques for each motor. If the motion of A and B are in the same direction, the load or torque will be the same on each and of similar sign. Hence as the load on the wheel C is increased, the speeds of A and B will tend to become equal, if A had been running faster than B, and for a certain load on C the speed of A will be equal to that of B. If the load on C is further increased, B will run quicker than A.

There will be a certain value for the load on C at which the motion of A will be reversed, and a further increase of the load on C will bring C to rest, A and B then rotating at equal speeds in opposite directions.

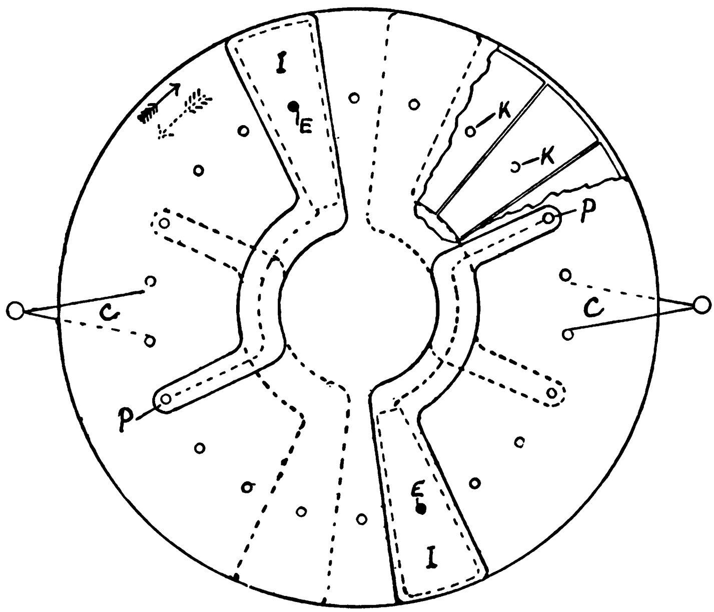
When the arrangement is such that the motors rotate in different directions when the load on C is nothing, suppose that A is running faster than B. The motion of C will depend upon the difference of the speeds of A and B. When a load is put on C, the motion of A is retarded, and that of B assisted, so that B takes less current and A takes more; the torques on the two motors, due to the load on C, now being of equal amount but of opposite sign. As the load on C is increased the speed of A will be reduced, and that of B increased until the two are equal and C comes to rest. B will now be acting as a generator, and sending a current on to A. If the load on C is simply that due to friction, the process cannot be carried any further. But if the load on C can be reversed, the speed of B will be greater than that of A, and the motion of C will be reversed.

This arrangement has been made use of in an electric steering-gear designed by the Union Electricitäts Gesellschaft. The intermediate wheel is made to actuate the rudder by a differential action, and the motion reversed by making the speed of one motor greater or less than that of the other.

XXVIII. *An Influence-Machine.* *By* W. R. Pidgeon, *M.A.**

On June 23rd, 1893, I showed an influence-machine at a Meeting of this Society, which was described in the Phil. Mag. for September 1893, but which had the disadvantage of being an expensive machine to make. I now show a form of machine which not only gives better results, but is both cheap to manufacture, and has qualities which may, I think, interest the Members of this Society, especially in regard to its suitability for exciting Röntgen-ray tubes.

The machine consists of one or more pairs of glass disks mounted on a spindle and running in opposite directions, with earthing-brushes arranged similarly to a Wimshurst machine. The disks are of ordinary glass, and are covered



with sectors about an inch or an inch and a half wide at the circumference, and placed about one-eighth of an inch apart.

* Read October 28, 1898.

These sectors stand radially, and each carries a small brass contact-knob K. The disks are covered with wax composed of half paraffin and half rosin by weight. The wax covers up and insulates the whole of each sector except the small brass contact-knob which peeps above it. Each of the earthing-brushes, E, passes through and supports a fixed insulated inductor, I, which is formed of tinfoil stuck to an ebonite backing and insulated with wax. The surface of the wax on all the inductors and disks is carefully varnished several times with filtered shellac to protect the wax and give it a hard surface. Each inductor is kept charged by a stationary point, P, connected to it and placed so as to collect, from the revolving disk, shortly before the main collecting brushes C. The sectors on each of the disks are earthed at the moment when they are passing between the opposite disk on the one hand and the fixed inductor on the other, both of which carry a charge of the same sign. The sectors therefore receive their charge at a moment when their capacity is at a maximum, owing to their standing between two charged inductors. As each sector moves away from the brush to the right and out of the influence of the inductor its capacity decreases, and therefore its potential rises, and when it is opposite the point at which its fellow disk is being earthed, its potential is proportionately much higher than in the Wimshurst form of machine; and it therefore induces a proportionately higher charge on the sector being earthed. This sector, as it moves away from its inductor to the left, again rises in potential; and on arriving at the earthing-brush induces a still higher potential on the sectors moving to the right. This cumulative action goes on in a sort of geometrical progression until, as a matter of fact, the output of this form of machine rises to about four times that of a Wimshurst of a similar size when measured by the overflow of a leyden-jar (25 oz.).

It may help to make the action of the machine more clear if we regard it from the point of view of its being a condenser, the plates of which can be charged in one position, then shuffled, to bring the positives and negatives together, and thus discharged. For the fact that each sector is imbedded in an insulator enables it to receive a charge on each face as it stands between the disk and the inductor, like a plate in a

condenser. It therefore carries forward a double charge, so to speak, as compared with that carried forward by a machine without inductors. Again, the drop of capacity and consequent rise in potential of the sector as it moves away from the inductor is so great, that the induction of the machine is also practically doubled, and hence the total output is multiplied by four. That is to say, a machine having inductors which act upon numerous insulated sectors is equivalent in output to four machines of the same size of the ordinary type.

W. R. Pidgeon's machine with one pair of plates—	
of 12 in. diam. requires 22 sq. ft. of area to pass the collecting-brushes per spark.	A 15-in. Wimshurst same day requires 64 sq. ft. per spark.
of 17 in. diam. do. 17·1 sq. ft. do.	A 15-in. do. 76 do. do.
18 in. do. 18·5 do.	A 15-in. do. 70 do. do.
19 in. do. 19·4 do.	A 15-in. do. 70 do. do.
2 pairs of 27 in. do. 16·4 do.	
Average ...... 17·85 sq. ft.	Average... 70 sq. ft.

Comparative efficiency nearly 4 to 1.

Mr. Wimshurst's 8-plate 15-in. machine, which he kindly tried for me himself, requires 97·07 sq. ft. per spark.

The length of spark between knobs is approximately the same as in a Wimshurst of the same size, but, if anything, slightly less. If, however, the fixed inductors are taken away and the wax disks run with the brushes on alone, the machine gives much longer sparks; but its output is then decreased to a little less than double that of a Wimshurst.

My machine, 19 in., without inductors, requires 45 sq. ft. per spark.

The wax which covers the disks prevents the sectors leaking from any point except the small brass contact-knobs, and thus enables the machine to work in the dampest weather; in fact, it may be sponged with water, or have water squirted at it, and yet will work if only it is first wiped up with a duster. Dirt, likewise, makes almost no difference; and usually the induction starts up before the disks have made a revolution, even though the machine may have been left standing for weeks. It will, moreover, work on short circuit;

and in all but the very worst weather, or after standing idle for a long time, it is not necessary for any of the brushes to actually touch the contact-knobs.

The fact that the only place from which a sector can leak electrically is from its small brass collecting-knob, makes it possible to run disks so large as to almost touch the first motion-shaft below or the collectors on either hand. This obviously allows the machine to be snugged up, and so save cupboard-room.

The collectors have been covered with ebonite, and everything has been done to expose as little naked surface as possible, partly for convenience in handling, but chiefly to enable the machine to be used in bad weather.

When exciting a Crookes or Jackson tube the knobs of the dischargers should be brought to either end of the tube, the terminals of which should also be capped with brass knobs to prevent any brush-discharge.

The tube should be of a sufficiently high resistance to use all the potential of the machine and not require a spark-gap; it should be roughly suitable for a coil-spark of 6 in. to 14 in., according to the size of machine with which it is excited.

The illumination produced will then be good and steady, and the tube may be run for an almost indefinite period without running the slightest danger of over-heating its terminals or of being troubled by its resistance changing.

A pair of 19-in. disks is adequate to show brightly the bones in the hand and arm, and, with some people, to faintly indicate the ribs on a screen; while a pair of 12-in. disks exciting a suitable tube is sufficient to show the hand- and wrist-bones clearly.

Discussion.

Capt. J. H. Thomson, R.A., said that, apart from its electrical merits, the machine possessed advantages in mechanical construction. He thought there was still room for improvement in this respect. The counter-shaft should be done away with, and ball-bearings should be introduced. The inductor was a distinct improvement; he thought the

efficiency might be increased by adding other inductors. Platinum-iridium was the best material for brushes of such machines.

Prof. AYRTON asked what efficiency was obtained with modern influence-machines in general.

Capt. THOMSON had found that when running a machine by a motor, about 80 per cent. of the power was wasted in mechanical friction; of the remaining 20 per cent. a great deal was lost as electrical leakage.

Prof. S. P. THOMPSON thought it had been pointed out by Mr. Wimshurst that influence-machines did not work well unless there were at least two thicknesses of glass between the inducing and induced conductors. That was why Mr. Wimshurst put his sectors on the outer faces of the glass disks. Mr. Pidgeon had departed from this. The advantage of the narrow spacing of the sectors was not very apparent.

Mr. WIMSHURST (abstract of communication). Waxing the disks reduces leakage and increases the output; the wax-coating virtually doubles the number of plates. Inductors contribute a further increase to the output. In 1883 Mr. Wimshurst tried thick coatings of shellac, and also duplicating the glass, with in some cases sectors upon the second glass to increase the capacity. The output was increased, but the construction lost simplicity. The indifference of Mr. Pidgeon's machine to dirt and dust was a most valuable result.

Mr. PIDGEON, in reply, showed a set of secondary inductors such as Capt. Thomson had just proposed. They improved the output by about 15 per cent., but they were troublesome to keep in order, for they increased the tendency to "reverse."

XXIX. *The Magnetic Fluxes in Meters and other Electrical Instruments.* *By* ALBERT CAMPBELL, *B.A.**

IN all electrical measuring instruments in which the deflecting or controlling forces are electromagnetic, the magnetic fluxes and fields are of great importance, and yet there seem to be no tables published which give even rough measurements of these, the result being that many people who are thoroughly expert in the use of instruments have no idea whatever of the order of magnitude of the magnetic fluxes occurring in the very commonest instruments. In order, therefore, to fill this gap to some extent, I have recently carried out a series of experiments on the subject, and although the list of instruments thus tested is not very extensive or complete, I have been able to include in it a good many of the more familiar types. As individual instruments of the same type vary somewhat amongst themselves, it would have been waste of time to have aimed at great accuracy in these measurements. Accordingly, whilst guarding against large errors in general, I have been content in one or two cases with results which only indicate the order of magnitude of the quantity measured.

In most cases the quantity determined has been B, the magnetic flux density or number of induction-tubes per square centimetre sometimes through iron, sometimes through air. In some cases Φ, the total flux, was measured, and for several of the meters determinations of the power lost in their various parts were also made.

Methods of measuring B.

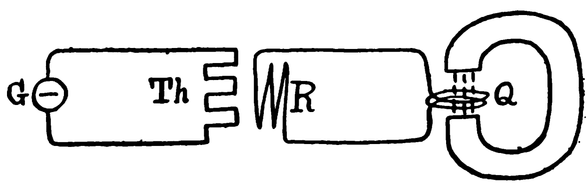
METHOD I. Except when the fluxes were alternating, the method used was the well-known way with a ballistic galvanometer. A search-coil in circuit with the galvanometer either had the flux suddenly reversed in it, or was pulled quickly out of the field. For many of the experiments small and

* Read October 28, 1898.

very thin search-coils had to be used. For instance, in one of the meters, the available air-gap was only 1 millim. across. The bobbins of these small coils were made by cementing together one or more round microscope cover-glasses between larger strips of mica for the ends; they were wound with from 10 to 200 turns of silk-covered copper wire of 0·075 millim. diameter. Much thicker wire was used when it was desirable to have the resistance of the galvanometer-circuit low. The smoothness of the mica cheeks allowed the coils to be withdrawn from position with the necessary quickness. The galvanometer was calibrated from time to time by means of a standard pair of coils whose mutual inductance was accurately known. As the measurement of small fluxes by the ballistic method presents no difficulty, and requires only ordinary instruments, further description is needless.

Method II. When, however, the small flux is an alternating one, the voltage set up in the search-coil is more difficult to measure; accordingly two special methods were here used. In the more accurate of these two methods

Fig. 1.



the search-coil Q (fig. 1), through which the flux was made to pass, instead of being in circuit with a ballistic galvanometer, was joined directly to a resistance-coil R, laid along one set of junctions of a minute thermopile Th, which last was connected with a sensitive galvanometer G. The resistance-wire was of manganin (silk-covered) and was placed along the junctions so as to be non-inductive, and to avoid producing eddy-currents. The thermopile consisted of ten pairs of thin iron and nickel wires each 7 millim. long. These metals were chosen as their thermo-electric lines are far apart, and almost parallel to one another.

Some years ago the writer showed (Proc. Roy. Soc. Edin. July 1887) that a thermopile used thus could give a fairly accurate measurement of the current through the resistance-wire, the ultimate deflexion being proportional to the square of this current. Hence, for a given frequency, the mean square of the P.D. at the terminals of the search-coil was proportional to the deflexion. Each time it was used the combination was calibrated in the following manner:—A measured current of 1 ampere from the alternating supply circuit used was passed through a non-inductive resistance of 0·2 ohm, and the resulting P.D. of 0·2 volt was applied to the ends of the resistance R. From the observed deflexion of the galvanometer the mean square of the volts per division was found *. In all cases when the search-coil was in circuit the frequency n was observed, being measured by a frequency-teller. If the resistance of the search-coil be negligible, and if the flux follows the sine law, we have

$$v_2 = 10^{-8} \times 2\pi n N_2 B s_2,$$

where

$$\begin{aligned} v_2 &= \text{voltage shown by galvanometer,} \\ n &= \text{frequency,} \\ N_2 &= \text{number of turns in search-coil,} \\ s_2 &= \text{area of search-coil,} \end{aligned}$$

and

$$B = \sqrt{\text{mean square } B}.$$

The other method, which was by means of a telephone, was a rougher way, and will be described below.

In Table I. are given some of the results obtained in the case of the simpler instruments, the third column giving the resistance of the instrument, the fourth column its full load, and the fifth the mean value of B at that load. As far as possible the positions from which the mean value of B were obtained were chosen so as to give an idea of the *working* flux-density, and except where otherwise stated the values given refer to full load.

* The thermopile method of measuring small voltages is now in use in the German Reichsanstalt.

TABLE I.

No.	Name.	Resistance, Ohms.	Full Load.	B (mean).
1.	Siemens Electrodynamometer......	0·58	4 amps.	80
	" "	0·0156	20 amps.	18
2.	Kelvin Ampere-Balance	0·58	10 amps.	65
3.	Bifilar Mirror Wattmeter (after Dr. Fleming).	0·0078	50 amps.	55
4.	Ayrton and Mather D'Arsonval Galvanometer.	324	......	450
5.	Weston Voltmeter	7·4	0·2 volt	870
6.	Davies Voltmeter (Muirhead) ...	481	3 volts	400
7.	Evershed Ammeter	0·00043	100 amps.	700
8.	Ayrton and Perry Magnifying Spring Voltmeter.	30	18 volts	14,200
9.	Richard Recording Ammeter	7×10^{-5}	500 amps.	580
10.	Dolivo Voltmeter	56	15 volts	75
11.	Nalder Voltmeter.....................	586	40 volts	70
12.	Any Tangent Galvanometer	......	Defl.=45°	0·26
13.	Kelvin Astatic Mirror Galvanometer.	13,000	......	0·008
14.	Evershed Ohmmeter, Old Type ...	......	(100 volts)	(about 0·2)
15.	" " New Type...	......	(200 volts)	(about 10)
16.	Campbell Frequency-Teller.........	(500)	(0·2 amp.)	280
17.	Bell Telephone (double pole)	126	......	3000
18.	Ayrton and Perry Variable Inductance Standard.	10·6	(1 amp.)	20
19.	Standard Inductance Coil (L=0·2 henry).	10	(1 amp.)	46

As the numbers in the above Table throw an interesting light on the behaviour of many of the instruments it seems desirable to discuss them more fully in order.

(1) *Siemens Electrodynamometer.*—Measurements taken at the middle and the top of the swinging coil (of 4 turns) in direction perpendicular to the plane of the fixed coils gave B=120 and 40 respectively for the thin coil, and B≑21 and 16 for the thick. It will be seen that when the thick coil is used the deflecting field is quite comparable with the earth's field.

This, of course, introduces an error with direct currents unless care is taken to place the instrument so that the direction of the earth's field is at right angles to that of the deflecting field, and in the proper sense, *i. e.* with the instrument looking east or west according to the direction of the current in the swinging coil. The above results show that

the maximum variation at 15 amps. introduced by wrong placing (viz., due to a field equal to twice that of the earth, or 0·36) would be about 2·5 per cent. of the mean deflecting field. This was verified by placing the electrodynamometer on a well-levelled turntable, and connecting its thick coil with a quite steady source of continuous current by means of twisted flexible leads. In Table II. are shown the values of the current indicated by the instrument when turned into various positions, the first column giving the direction towards which the front was turned in each case.

TABLE II.

Direction.	Apparent Amperes.	Variation from Minimum.
N. . . .	15·00	0
W. . . .	15·20	1·3 %
S. . . .	15·32	2·1 %
E. . . .	15·20	1·3 %

It will be seen that the extreme variation is 2·1 per cent., which agrees (within the limits of error of the instrument) with the 2·5 per cent. variation deduced from the observed fields. For lower currents the variation is much more, being in the inverse ratio of the current. With the thin coil the field due to the fixed coil is so much stronger that the variation is slight except at the very lowest currents.

The thick coil had 7 turns and an area of about 46 sq. centim., and the total maximum flux at full load was found to be 1850, giving mean $B \doteqdot 40$. At full load the power wasted is 6·2 watts and 9·3 watts for the thick and thin coils respectively.

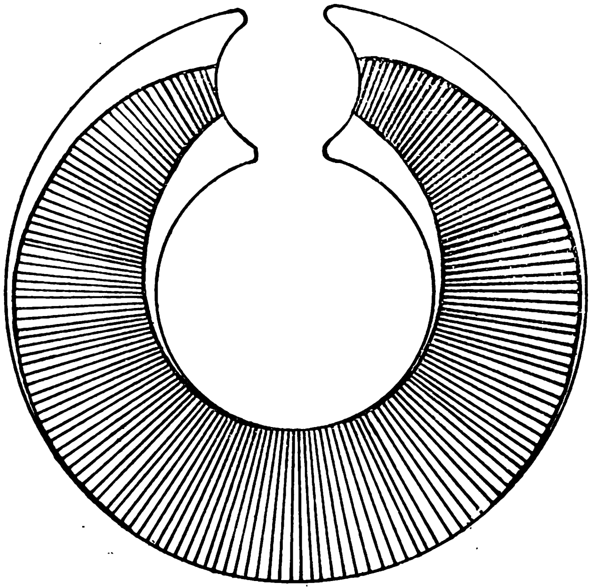
(2) *Kelvin Balance.*—The total flux through the central space of the coils was got by winding the search-coil round the supporting pillar, and taking throws by reversing the current. The resultant flux was about 1600 (for full load). By the astatic arrangement of the swinging coils the instrument is made independent of the earth's field. The self-inductance is about 0·0016 henry.

(3) *Bifilar Mirror Wattmeter.*—This instrument has ranges up to 50 or 100 amperes at 10 volts and upwards. The

numbers given refer to the fixed series-coil. With direct currents it is clear that precautions have to be taken to eliminate the effects of the earth's field.

(4) *D'Arsonval Galvanometer.*—This was a ballistic one (made by Paul) with a narrow swinging coil of the Ayrton-Mather type. The B given in the Table is that in the air-gap in the neighbourhood of the moving coil; it would seem to be sufficiently great to be practically unaffected by the magnetism of the earth. Besides, as it is an instrument for use in a fixed position, it is only the effect of variable external fields that need to be taken into account. This point will be discussed a little further on. To get an idea of how the flux in the steel varies from point to point along the annular magnet, an experiment was made with a ring-magnet of rectangular section, having an air-gap as shown in fig. 2.

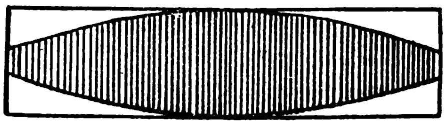
Fig. 2.



A small search-coil which could only just slip along the magnet was moved by jerks into successive positions, and the corresponding changes in the flux were calculated from the throws on a ballistic galvanometer in circuit with the coil.

Fig. 2 shows the result, the radial breadth of the shaded part being drawn proportional to the flux in the steel at each position. Fig. 3 is a similar diagram for an ordinary bar-magnet. In the ring-magnet the available air-gap flux was less than one third of the maximum flux.

Fig. 3.



(5) *Weston Voltmeter.*—It will be noticed that B in the air-gap of this instrument is very high, viz. 870. This might lead one to suppose that the earth's field would have no perceptible effect on its readings, but it must be remembered that the flux induced in a piece of iron or steel in the earth's field is usually very many times greater than the flux in air due to the earth alone. This can be easily shown by connecting a coil with a ballistic galvanometer and reversing the coil with regard to the earth's field first by itself and then with a soft iron core in it. The throws of the galvanometer will be enormously increased by the presence of the core. To find how far the earth's field affected the flux in a permanent magnet with a moderate air-gap, a coil was wound upon the circular one shown in fig. 2, and was connected with a galvanometer. The magnet was then turned round so as to quickly reverse the action of the earth's horizontal field upon it. The resulting throw on the galvanometer showed that the maximum B in the steel, which was about 5000, was only changed by 3 lines per sq. cm., *i. e.* by less than 0·1 per cent. The behaviour of the Weston magnet tallies with this, for when the instrument, with a steady voltage on its terminals, was turned round to face each point of the compass, no change in the reading could be detected, although the scale could be read to about 1 in 1000*.

(6) *Davies Voltmeter.*—In this instrument one side of the rectangular moving coil moves in a narrow cylindrical air-

* At Professor Ayrton's suggestion I have re-tested the instrument at the higher readings, and have detected a variation of about 0·1 per cent.

gap between specially shaped pole-pieces of a strong permanent magnet. It gives a maximum angular deflexion of about 210°.

(7) *Evershed Ammeter.*—A coil of 6 turns magnetizes two small pieces of iron with a movable piece between them. B was measured at about the end of one of the fixed pieces where the movable piece faces it.

(8) *Ayrton and Perry Magnifying Spring Voltmeter.*—The small iron tube, which is surrounded by a coil and pulled down by it, is 7·2 centim. long, and the iron has a section of about 0·12 sq. centim. The number given in the table is the average B *in the iron* for the whole length of the tube for a load of 15 volts, *i. e.* 0·8 of fall load.

(9) *Richard Recording Ammeter.*—This has two solid iron cores (each 4 sq. centims. cross-section) round which the current is carried by a single turn of copper strip ; an iron armature carrying the pointer is attracted by these cores. B was measured between one core and the armature. It will be seen that the total flux is large ; thus a strong deflecting force is obtained which makes the friction of the recording pen of less account, but on the other hand much error from hysteresis comes in.

(10) *Dolivo Voltmeter.*—Here a thin wire of soft iron is drawn down into a solenoid. The value given is for the hollow core of the coil (including the wire).

(11) *Nalder Voltmeter.*—In this a small piece of soft iron moves in the magnetic field produced by a coil outside it. The number in the table is for the middle of the space within the coil.

(12). *Tangent Galvanometer.*—When the deflexion $=45°$, the resultant field $=$earth's field$\div\cos 45°$

$$\doteqdot 0{\cdot}26 \text{ (in London).}$$

(13) *Kelvin Astatic Mirror Galvanometer.*—The galvanometer was made very sensitive and almost unstable by means of the controlling magnet. The mean control field (for this condition) was found by measuring its sensitivity and comparing it with that when the earth's field alone was used. The deflecting field for 1° would be less than 0·00002, and depends on the degree of astaticism of the suspended magnets.

(14) *Evershed Ohmmeter*—(Old type; polarized, astatic). The number given is a rough approximation to the B due to the shunt-coil alone at 100 volts. About the middle of the scale the field due to the series-coil would have the same value.

(15) *Evershed Ohmmeter*—(New type, with soft iron needle). The B given is that due to the shunt-coil alone (at 200 volts). It is clear that the earth's field cannot introduce much error. In any case the errors due to external fields can be eliminated (as the makers direct) by reversing the current and taking the mean of the readings.

(16) *Campbell Frequency-Teller.*—The B is measured between the vibrating strip and the attracting pole of the electromagnet. It will be seen that only a quite moderate field is required to throw the spring into strong vibration when it is adjusted to the right length for resonance.

(17) *Double Pole Bell Telephone.*—The diaphragm was so close to the poles of the permanent magnet as to form an almost closed magnetic circuit. A small search-coil was wound round one of the pole-pieces (area $=0{\cdot}24$ sq. centim.), and the diaphragm was then laid in its place. A throw of the galvanometer was got by pulling off both the diaphragm and the search-coil. Thus the flux-density given refers to the pole-pieces.

(18) *Ayrton and Perry Variable Standard of Self-Inductance.*—With the pointer at 0·038 henry, the total flux within the inner wooden bobbin was about 5200 for 1 ampere.

(19) *Self-Inductance Standard* ($L=0{\cdot}20$ henry).—This was a coil of 1158 turns of insulated copper wire (diameter 1·24 millim.), the outer diameter of the coil being 22 centim. and the height 9 centim. The B given in the table is the average for the whole cross-section of the coil at the middle of its height. The total flux corresponding to this was 17,500. B at the centre of this cross-section was 53.

In addition to the instruments already discussed, experiments were also made upon a number of meters of different types, some being for direct, and some for alternating currents. As some of the types vary in construction from year to year, a few words of description in each case will make the results clearer.

Aron Watt-hour Meter (1894 type).—Range to 50 amps. at 100 volts. Two pendulums, each carrying shunt-coils, are acted on by series-coils under them, one being accelerated and the other retarded. With both series and shunt-currents passing (at full load), the mean B between the fixed and movable coils was about 70.

Frager Watt-hour Meter.—Range to 10 amps. at 100 volts. A meter of the "Feeler" type, in which the shunt and series coils form an ordinary wattmeter, whose deflexions are integrated at intervals. The mean B was measured as near the centre of the shunt-coil as possible.

At full load	B = 63,
With shunt-current alone . .	B = 13,
With series-current alone . .	B = 50.

Hookham Direct-Current Ampere-hour Meter.—Range to 100 amps. A small disk, surrounded by mercury which carries the current, is cut twice by part of the magnetic flux from a strong permanent magnet (of cross-section 7·5 sq. centim.). The disk is thus caused to turn. On the same spindle is a brake disk of copper (5·4 centim. diameter), which is also cut by a part of the flux from the same permanent magnet. Unfortunately it was not possible to take the meter to pieces, so the driving flux could not be measured. By slipping a search-coil along the permanent magnet, the total leakage was found to be over 26,000 lines. The total brake flux passes through the disk at four air-gaps, two and two in series, and has a value of about 5000 lines. The 8 pole-pieces which direct the flux have cross-sections of 1·53 centim. each. In two of them the iron near the air-gap is turned down so as to leave only a thin neck of about 0·12 sq. centim. cross-section. This is supposed to increase the permanence of the flux. At these necks more than $\frac{1}{4}$ of the total flux leaks from the iron. Whether they increase or diminish the permanence seems quite uncertain. The mean flux density at the four air-gaps was found to be about 1020.

The power spent in the meter at full load = 12·8 watts.

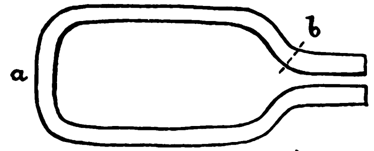
The power spent in turning the spindle (at 2·2 revolutions per second) with full load was measured by the method

(1) described below, and was found to be 0·016 watt*. Hence the efficiency of the meter *as a motor*=0·125 per cent.

Kelvin Ampere-hour Meter.—Range to 600 amperes. In this a thin iron core, kept highly magnetized by a shunt-current, is drawn down into a solenoid which carries the main current. The solenoid had 6 turns (*i. e.* 3600 ampere-turns at full load), and was about 16 centim. long. At full load the flux density at the lower end of the solenoid was over 250. With the shunt-current alone the total flux through the moving coil and core was about 330.

Elihu Thomson Watt-hour Meter.—Range to 50 amperes at 100 volts. This meter consists of a small ironless motor, in which the series-current goes through the field-magnet coils and the shunt-current through the armature. On the armature spindle is a brake disk of copper (13·5 centim. diameter), which passes between the narrow air-gaps of three permanent magnets of the shape shown in fig. 4. These magnets are often of different strengths, being chosen to give the proper brake-force for each individual meter. Their polar faces are

Fig. 4.



about 7·5 sq. centim. The mean B in the air-gaps was about 700. By the method of placing the magnets the greater part of the flux acts on the brake disk. By slipping a search-coil along one of the magnets it was found that the total flux at *a* was about 15000, making B about 7000. Of this flux nearly one half remains in the steel as far as the section at *b*.

Driving Flux.—Without the shunt-current the full load current gives mean B $\doteq$ 130 along the axis of the series-coils. The shunt-current at 100 volts gives a field at right angles

* After the author had measured the motor-efficiencies of several meters, Mr. Sidney Evershed somewhat anticipated him by announcing (Institution of Electrical Engineers, May 12th, 1898) one or two similar results, making no mention, however, of the method by which the results were obtained.

to this in which B = 10. The shunt-current also passes through a "compounding" coil fixed coaxially in one of the series-coils for the purpose of overcoming friction at the lower loads. The B due to this starting coil alone is about 3. There is a small stray field from the series-coils perceptible at the brake disk, but as it is less than the $\frac{1}{300}$ part of the field due to the permanent magnets its effect may be neglected.

Effect of the Earth's Field.—Since the driving flux density at full load is only 130, it is clear that at the lower loads the rate of the meter may be considerably affected by the earth's field. To test this point the meter was levelled up on a turn-table (as in the case of the electrodynamometer already described), and a constant load of 4·625 amperes at 100 volts was kept on it. The load was measured by a Kelvin balance and a reflecting multicellular voltmeter. The rate of the meter, *i. e.* spindle revolutions per watt-second, was then determined

(A) with alternating current;
(B) with direct current, the earth's field helping the driving field;
(C) the same, with the earth's field opposing the driving field.

Table III. shows the results of the tests.

TABLE III.

Position of Meter.	Current.	Rate.
(A) Facing East.	Alternating at 80~ per second.	0·0002125
(B) „ West.	Direct.	0·0002135
(C) „ East.	Direct.	0·0002212

It will be seen from (B) and (C) that by turning the meter round through 180° its rate at $\frac{1}{10}$ load is altered by 3·6 per cent., which is exactly what might have been predicted from the value of B given above. The rate with alternating current does not lie between the rates (B) and (C) as it ought, but is about 0·5 per cent. slower than (B).

Power Spent in Meter.—The resistance of the series-coil $\doteqdot$ 0·0066 ohm, and that of the shunt-coil $\doteqdot$ 2030 ohms; hence the power spent in heating the coils $=16·5+4·9=21·4$ watts.

The power spent in actually driving the meter (at full load) was measured by two methods as follows:—

(1) An arm of about 10 centim. long was attached to the spindle of the meter (at right angles to it). With full load switched on, the tangential force f necessary to hold at rest the end of this arm was measured by the extension of a spiral spring which had been calibrated by known weights.

Then

$$\text{Power (in watts)} = 10^{-7} . f . 2\pi r . \frac{n}{t},$$

where

r = length of arm,

$\frac{n}{t}$ = number of revolutions per second when the spindle is free to move.

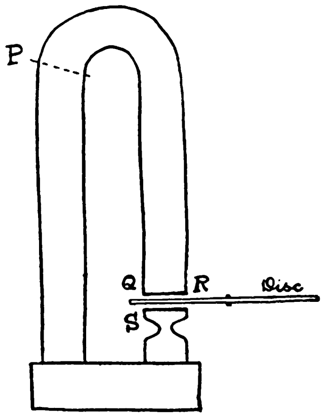
(2) In the second method the shunt-circuit was disconnected and joined directly to a sensitive galvanometer with a resistance of 12,000 ohms in circuit. A measured current was sent through the series-coils, and the spindle was turned at about the full rate. Thus the meter acted as a dynamo and gave a deflexion on the galvanometer. By watching the galvanometer-scale it was not hard to keep this deflexion steady, and from the known calibration of the galvanometer the voltage given by the armature was obtained. The total E.M.F. multiplied by the normal shunt-current (0·0493 amp.) gave the driving-power.

Method (1) gave for the driving-power at full load 0·020 watt, whilst method (2) gave 0·021 watt. Hence the efficiency of the meter as a motor is only 0·095 per cent.

Hookham Alternating-Current Watt-hour Meter.—Range to 10 amperes at 100 volts. In this meter a solid iron core forming an almost closed magnetic circuit is magnetized by a shunt-coil, which latter, by reason of its large inductance, carries a current which lags behind the main current by 50° to 60°. A smaller U-shaped piece magnetized by the main current has its poles close to the upper pole of the

larger iron core, and one of the poles carries a copper screen. A small aluminium disk (8·8 centim. in diameter), partly between the poles, is turned by the rotary magnetic flux thus produced. The brake force acts on the same disk, and is due to a tall permanent magnet about 20 centim. high (with a narrow air-gap), as shown in fig. 5.

Fig. 5.

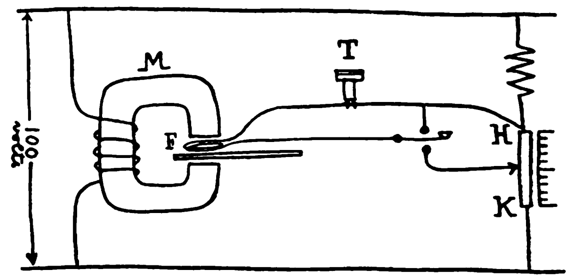


Brake Flux.—The maximum flux in this magnet was found to be near P, and had the value 19,820, corresponding to B=8800. Of this only 4400 lines ultimately cut the disk, much being lost by cross leakage from Q and S to the opposite limb. Thus less than $\frac{1}{4}$ of the maximum flux is made use of. The mean B between the poles was found to be 650.

Driving Flux.—The distribution of the somewhat complicated alternating field was traced by the following method, which also gave rough quantitative results. A telephone T (fig. 6) was arranged in a circuit with the search-coil F and the low-resistance strip H K in such a way that the strip could be switched out of circuit at will. One of the connexions to H F was through a sliding contact, so that the resistance of the part in the telephone-circuit could be varied from 0·05 ohm downwards. A current of 1 ampere was maintained in the strip and was from the alternating circuit which supplied the meter load. With the strip out of circuit

the search-coil was moved into various positions, and the distribution of the alternating flux observed by means of the sound in the telephone. To measure the flux at any position H K was set to such a value that the small P.D. introduced by it into the telephone-circuit gave a sound of the same loudness as that given when the search-coil was placed in the flux. The absolute values given by this method came out 10 to 15 per cent. too small; but it proved a very convenient way

Fig. 6.



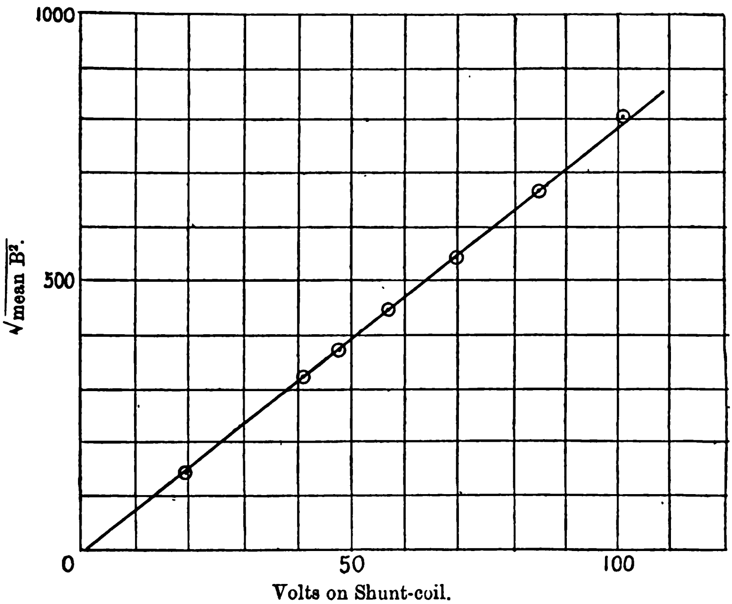
of comparing the flux-densities at various positions. It was thus found that when the main current (full load) was switched on in addition to the shunt-current, the flux-density under the middle pole was nearly doubled. The most curious fact brought to light by the method was that across the air-gap of the permanent magnet a considerable alternating flux is induced, the value of B at that position being actually about $\frac{1}{4}$ of that between the poles of the shunt-magnet with the shunt-current alone. The permanent magnet forms a kind of secondary magnetic circuit directing around itself the alternating eddy-currents in the disk. Whether this has any sensible demagnetizing effect upon the permanent magnet the writer has not determined.

The fluxes were measured more exactly by the method of the search-coil and thermopile described above. With shunt alone the root of mean square B was about 50 in the air-gap and 800 in the iron core just above the shunt-bobbin. The shunt-current at 100 volts (86 $\sim$ per sec.) was found to be 0·031 ampere. If the current followed a sine curve its maximum value would be 0·044. When a direct current of this

value was tried the fluxes produced were much larger than those with the (supposed) equivalent alternating current. This is partly due to the fact that the shunt-current does not follow the sine law, but is no doubt also due to the existence of eddy-currents in the iron core. That these currents even in the disk spread the flux and reduce the flux-density in the air-gap was shown qualitatively by placing a search-coil, connected with a telephone, in the air-gap of a ring electromagnet excited by alternating current. The sound in the telephone was lessened when a copper disk was held near the coil in the air-gap. The search-coil and thermopile method showed that the flux-density had been reduced by 8 per cent.

To find how the core-flux varied with alteration of the voltage on the shunt-coil, by the same method the B just above the shunt-bobbin was measured for a number of voltages

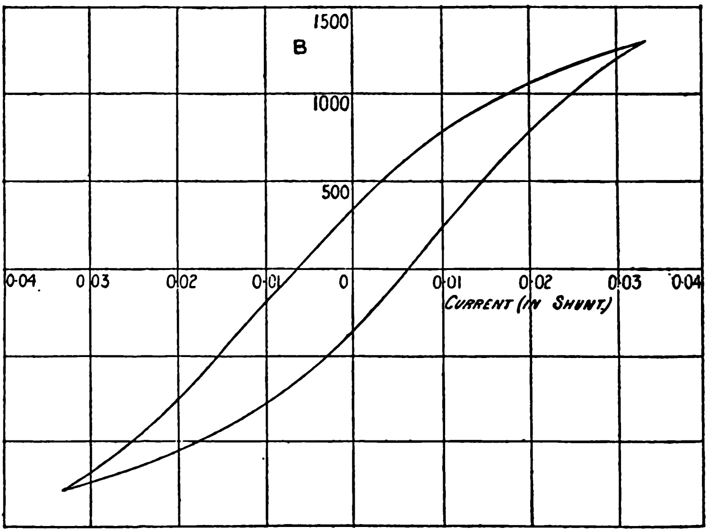
Fig. 7.



from 20 up to 100 volts. It will be seen from the curve in fig. 7 that B is very nearly proportional to the potential-difference. In practice it is found that the speed of rotation is very nearly proportional to the voltage.

Power spent in Meter.—To approximate to the amount of power spent by reason of hysteresis when the shunt-current alone is on, the iron was carried through a cycle by means of direct current of such amount as to give a maximum B nearly corresponding to that given with alternating current at 100 volts. The curve obtained is shown in fig. 8.

Fig. 8.



From this curve it is not possible to get the exact value of the hysteresis loss, as the iron core is not a uniformly magnetized closed circuit. To get an idea, however, of the amount of the power wasted by hysteresis, let us suppose that the magnetic circuit is equivalent to a uniform iron ring of the same cross-section as the core of the shunt-coil, uniformly wound with the same number of turns n_1 as the shunt-coil * (carrying the same current), and traversed by a flux equal to

* The exact value of n_1 was not known, but the value (1200) used was estimated from the resistance and gauge of the shunt-wire and the size of the coil.

that at the part of the meter-core for which the curve in fig. 6 was taken.

Let $c =$ current at any moment,

and $s =$ section of ring.

Then

$$\begin{aligned}\text{Hysteresis-loss in joules per cycle}\\ &= \int c\,dv = 10^{-8}n_1\int c\,d\Phi = 10^{-8}n_1 s\int c\,dB\\ &= 10^{-8}n_1 s \times (\text{area of curve})\\ &= 10^{-8} \times 1200 \times 3\cdot7 \times \text{area}\\ &\doteqdot 0\cdot00114.\end{aligned}$$

$\therefore$ at 86$\sim$ per second,

$$\text{Power spent} \doteqdot 86 \times 0\cdot00114 = 0\cdot098 \text{ watt.}$$

It was found by direct measurement that the actual power spent in the meter (with the shunt-current alone) was far larger than this. Accordingly measurements were made of the rise of temperature of the iron core by means of an iron-nickel thermopile. Three junctions of the pile were bound against the iron, which had its surface well insulated with paint. A pad of wadding was tied over the spot, and the thermopile was connected with a suitable galvanometer. The shunt-current was switched on for 120 seconds, the deflexion being read at intervals, and on breaking the current the cooling was observed for several minutes. The thermopile and galvanometer were calibrated with a known difference of temperature, and the curves of heating and cooling were drawn. The curve of heating was then corrected by means of the other curve, and thus the heating of the iron (corrected for cooling) was found.

At a spot just above the shunt-bobbin the corrected rise of temperature was $0^\circ{\cdot}87$ C., whilst near the air-gap it was only about half of this. Taking account of this last fact, we may take that a volume of about 61 c.c. was raised in temperature by $0^\circ{\cdot}87$ C.

Now

$$\text{Power spent} = 4\cdot2\frac{\sigma . \text{VD} . \partial\text{T}}{t} \text{ watts,}$$

where σ = sp. heat of the iron,
V = volume " "
D = density " "
∂T = temperature-rise (corrected),
t = time in seconds;

therefore from the results above,

$$\text{Power} \doteq 1{\cdot}58 \text{ watts.}$$

Since the hysteresis-loss is about 0·10 watt, it will be seen that the eddy-current loss $\doteq$ 1·48 watts.

The power lost by eddy-currents in the disk was similarly measured, and was found to be about 0·02 watt.

In the shunt-coil the C^2R loss = 0·57 watt; and hence the total copper and iron losses = 2·17 watts.

Direct Measurement of Total Power.—The total power given to the meter was measured by the three-voltmeter method, in which was used a reflecting electrostatic voltmeter accurate to about 1 in 1000 at all the points of the scale required. The result obtained was 2·06 watts, which agrees fairly well with the total watts shown by the other methods.

Driving-Power.—The driving-power at full load was also measured by the method of the spring-balance described above; it was found to be 0·00073 watt. Taking account of the series-coil (whose resistance was 0·009 ohm) the total power spent at full load is over 3 watts; and hence

$$\text{Motor efficiency} \doteq 0{\cdot}024 \text{ per cent.}$$

Scheefer Alternating-Current Watt-hour Meter.—Range to 10 amperes at 200 volts. In this meter a pile of E-shaped iron stampings has the series-coil on one outer limb and the shunt-coil on the other. An aluminium cylinder is turned by the rotary field thus produced, and on the same spindle is a brake disk with a single magnet exactly like those in the Elihu Thomson meter.

The mean B between the poles of this magnet was about 480. With the shunt-current alone the $\sqrt{\text{mean } B^2}$ between the shunt-pole and the driving cylinder was 570. The total

power spent with the shunt alone on is 11·5 watts, of which 7·1 watts are due to C^2R loss.

Shallenberger Alternating-Current Ampere-hour Meter.—Range to 20 amperes. A series-coil (of about 50 turns) and a small short-circuited secondary coil, with their axes at an angle of about 45°, produce a rotary field by which is turned a small disk with a soft iron rim. The brake-force is obtained by air-friction on four aluminium vanes.

By the search-coil and thermopile method it was found that inside the series-coil

$$\sqrt{\text{mean } B^2} \doteqdot 100.$$

Power spent in Meter.—The resistance of the series-coil was 0·025 ohm; hence the power spent in it $=10·0$ watts.

The power spent in the copper stampings which form the secondary coil was found by measuring their rise of temperature with a small copper-iron junction. This rise (at full load) was found to be about 0°·33 C. per minute, the cooling being negligible. The volume of copper was about 21·4 c.c.; whence the power spent $=0·40$ watt.

The driving-power was found to be 0·0069 watt; therefore the motor efficiency $=0·066$ per cent.

Current in Secondary Coil.—The current in the short-circuited secondary coil could not be measured directly. Calculating from the dimensions of the coil, however, the resistance was found to be $8·5 \times 10^{-6}$ ohm. From this and the value of the power (0·40 watt) we find that the secondary current attains the extraordinary value of 220 amperes.

For the sake of comparison some of the above results are collected in Table IV.

TABLE IV.

Name.	Driving B.	Brake B.	Power spent.	Motor Efficiency.
			watts.	per cent.
Elihu Thomson	130	700	21·4	0·095
Hookham (direct curr.) ...	(not measured)	1020	12·8	0·125
Hookham (alternat. curr.)	50	650	3·2	0·024
Shallenberger	100	......	10·4	0·066

In conclusion, it will be noticed that in motor meters the driving-flux density is of the order 100 and the brake B from 500 to 1000; also that the motor efficiencies are all very small, particularly in the case of the alternating-current meters. In all of them the greater part of the power taken is spent in heating conductors (either by eddy-currents or otherwise). If a small fraction of this wasted energy could be employed to overcome with certainty the friction at the lowest loads, a great advantage would be gained thereby.

June 7th, 1898.

Discussion.

Prof. Ayrton said he wished to offer some remarks on behalf of Mr. Mather and himself. The paper would perhaps have received more adequate discussion at the Institution of Electrical Engineers, for it was chiefly of a technical character. The importance of neutralizing the effect of leads when using instruments with very weak fields, such as Siemens's electro-dynamometer, should be emphasized. In instruments of the Kelvin-balance type, where two opposed coils carry two opposed currents, the field spreads at the edges; the true "working" flux is not that directly between the coils. Mr. Campbell would have done better if he had used a long search-coil wound round one of the swinging coils, forming part of a vertical cylinder. It would have been well also to have supplied some experimental proof that the astatic arrangement of the swinging coils of the Kelvin balance makes the instrument independent of the earth's field. The effect of the earth's field is of the order 0·2, so that with instruments of the Weston type, with a field of the order 1000, it was sometimes assumed, erroneously, that the readings were practically independent of the earth's H. Prof. Ayrton's own tests showed that by turning a Weston voltmeter towards different points of the compass, the errors in a particular case were far greater than might be predicted from the above ratio; the induction in the voltmeter pole-space, due to the earth's field, was much higher

than 0·2, the earth's field being exaggerated by the iron pole-pieces; it was not necessary to suppose that the magnetism of the permanent magnet caused the variation. The error observed was about 0·2 per cent. in a horizontal field, and 0·8 per cent. when the field of the voltmeter was parallel to the earth's induction. Here the induction in the gap was 1200, and H=0·2. In tests relating to the Ayrton and Perry magnifying-spring voltmeter, it was more important to know the B in the air-space *near* the iron than the B *within* the iron. Eddy currents might account for the extraordinary results obtained in the Shallenberger meter.

Mr. J. H. Reeves described a method he had adopted for measuring the effect of stray fields upon ammeters and voltmeters. The instrument to be tested is first mounted on a stand, and is brought under the influence of a large coil carrying a current. In this way fields of known magnitude can be superimposed on the working field throughout the range of the instrument, and the change of deflection due to them can be observed. From these known values the working field can be deduced. For let the current in the solenoid of the instrument at any moment be A amperes, producing a corresponding unknown working field of magnetic force X. Then X is proportional to the solenoid current, as measured by the indications of the instrument. If a magnetic force x is superimposed on X, then x is measured by x/X of A. If x is known, the working field X can be calculated from the change of deflection produced by the superposition. With Evershed ammeters the field measured in this way was in one instrument 200, and in another 226; or about one-third of Mr. Campbell's figure (700) for the Evershed ammmeter. Mr. Campbell's value of B did not represent the working field, but the field at the end of one of the fixed pieces of iron.

Mr. Campbell, in reply, said he thought the theory of electrical instruments to be well within the limits of Physics, and he had for that reason presented the paper to the Physical Society. The position chosen for the search-coil in the

Kelvin-balance tests may not have corresponded to the working flux, but it was near to the right position, and he had carefully specified the position chosen. His results as regards the Weston instrument differed from those of Prof. Ayrton; the errors he had observed for the particular voltmeter used were under 0·1 per cent. The earth's field probably produced an effect different for different Weston instruments, according to the degree of saturation of the permanent magnet. In his tests the Weston instrument did not have an iron case.

XXX. *On the Propagation of Damped Electrical Oscillations along Parallel Wires.* By Prof. W. B. MORTON, *M.A.**

IN a paper published in the 'Proceedings' of the Physical Society (*ante*, p. 219), Dr. E. H. Barton has compared the attenuation of electrical waves in their passage along parallel wires, as experimentally determined by him, with the formula given by Mr. Heaviside in his theory of long waves. The results show a large discrepancy between the theory and the experiments, the observed value of the attenuation constant being about twice too large. Dr. Barton discusses several possible causes of error and finds them inadequate, and suggests that the reason of the difference may lie either in (1) the nearness of the wires to one another, or (2) in the damping of the wave-train propagated by the oscillator. To these may be added (3) the consideration that the formulæ used were deduced by Mr. Heaviside from the discussion of his "distortionless circuit," in which the matter is simplified by supposing sufficient leakage to counteract the distortion produced by the resistance of the leads, whereas in Dr. Barton's circuit the leakage was negligible.

It is probable that the nearness of the wires has an appreciable effect on the phenomenon. The discrepancy would be diminished if the actual resistance of the wires was greater than that calculated by Dr. Barton from Lord Rayleigh's high-frequency formula. Now the effect of the neighbourhood of two wires carrying rapidly oscillating currents in opposite directions is to make the currents concentrate towards the inner sides of the wires †; and this would cause an increase in the effective resistance.

I have examined the effects of (2) and (3), viz. of the damping and the want of balance in the constants of the circuit. The investigation is perhaps of some interest owing to the fact that these elements are always present in the

* Read November 11, 1898.

† *Cf.* J. J. Thomson, 'Recent Researches,' p. 511.

ordinary experimental conditions; although, as will be seen, we are led to the conclusion that in all actual cases their influence on the phenomena is of quite negligible order. The method is the same as that used by Mr. Heaviside.

General Theory.—Let the inductance of the circuit be L, its capacity S, its resistance (of double wires) R, and its leakage-conductance K, all per unit length. An important part is played by the ratios $\frac{R}{L}$ and $\frac{K}{S}$; we shall call these ρ and σ. When ρ and σ are equal we have the "distortionless" circuit above referred to.

Now if V be the difference of potential between the wires and C the current in the positive wire, we have the equations

$$-\frac{dV}{dz}=\left(R+L\frac{d}{dt}\right)C, \quad \ldots \ldots \quad (1)$$

$$-\frac{dC}{dz}=\left(K+S\frac{d}{dt}\right)V; \quad \ldots \ldots \quad (2)$$

giving

$$\frac{d^2V}{dz^2}=\left(R+L\frac{d}{dt}\right)\left(K+S\frac{d}{dt}\right)V$$

$$=\frac{1}{v^2}\left(\rho+\frac{d}{dt}\right)\left(\sigma+\frac{d}{dt}\right)V, \quad \ldots \quad (3)$$

since $LSv^2=1$, where v is the velocity of radiation.

To simplify the algebra we shall work first with $V=V_0e^{-mz+nt}$, which can be made to represent damped periodic vibrations by giving complex values to m and n.

The equation (3) now becomes

$$m^2v^2=(\rho+n)(\sigma+n); \quad \ldots \ldots \quad (4)$$

and the connexion between C and V is given by (1), viz.

$$C=\frac{mV}{R+Ln}. \quad \ldots \ldots \ldots \quad (5)$$

To find the effect of a pure resistance R′ between the ends of the wires, as in Dr. Barton's experiments, put V_1, V_2, C_1, C_2 for the potentials and currents in the incident and reflected waves respectively. Then we have

$$C_1=\frac{mV_1}{R+Ln}, \quad C_2=\frac{-mV_2}{R+Ln};$$

also the total potential-difference V_1+V_2 is connected with total current C_1+C_2 through resistance R′ by Ohm's law,

$$V_1+V_2=R'(C_1+C_2).$$

These equations give for the reflexion factor

$$\frac{V_2}{V_1}=-\frac{R+Ln-mR'}{R+Ln+mR'}. \quad . \quad . \quad . \quad . \qquad (6)$$

If the circuit be distortionless and $R'=Lv$, then, as Mr. Heaviside showed, the absorption of the waves by the terminal resistance will be complete. We may regard this as the critical resistance for the circuit, and we shall express R′ in terms of it by putting $R'=xLv$. We then have

$$\frac{V_2}{V_1}=-\frac{\rho+n-mvx}{\rho+n+mvx}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Damped Wave-Train.—To pass to the case of a damped train transmitted from the origin in the positive direction of z we put

$$m=-\beta+i\alpha, \qquad n=-q+ip.$$

The difference of potential between the wires at any point after the head of the wave-train has reached this point is then represented by an expression of the form

$$V_0e^{\beta z-qt}\sin(pt-\alpha z).$$

The velocity of propagation is $\frac{p}{\alpha}$, the frequency $\frac{p}{2\pi}$, the logarithmic decrement $\frac{2\pi q}{p}$. If the waves suffered no attenuation in their passage along the leads we should have

$$\beta z-qt=0 \text{ when } z=\frac{pt}{\alpha}, \text{ i. e. } \beta=\frac{q\alpha}{p}.$$

In general, it is plain that $\left(\frac{q\alpha}{p}-\beta\right)$ measures the attenuation.

Inserting the complex variables in equation (4) we have

$$v^2(\beta-i\alpha)^2=(\rho-q+ip)(\sigma-q+ip);$$

$$\therefore \quad v^2(\beta^2-\alpha^2)=q^2-p^2-q(\rho+\sigma)+\rho\sigma; \quad . \quad . \quad . \quad . \qquad (8)$$

and

$$2v^2\alpha\beta=p(2q-\rho-\sigma); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

whence

$$v^2(\beta^2+\alpha^2)=\sqrt{\{p^2+(q-\rho)^2\}\{p^2+(q-\sigma)^2\}}. \quad . \qquad (10)$$

Velocity of Propagation and Attenuation.—In actual cases ρ and σ are small compared with p. If the damping is considerable, q may be comparable with p. Accordingly we expand the right-hand side of (10) in ascending powers of ρ and σ and solve for $v\alpha$ and $v\beta$. As far as terms of the third order in ρ and σ we find

$$v^2\alpha = p + \frac{p(\rho-\sigma)^2}{8(p^2+q^2)} + \frac{pq(\rho+\sigma)(\rho-\sigma)^2}{8(p^2+q^2)^2} + \ldots, \quad . \quad (11)$$

$$v\beta = q - \tfrac{1}{2}(\rho+\sigma) - \frac{q(\rho-\sigma)^2}{8(p^2+q^2)} + \frac{(p^2-q^2)(\rho+\sigma)(\rho-\sigma)^2}{16(p^2+q^2)^2} + \ldots . \quad (12)$$

Hence the *velocity of propagation*

$$= \frac{p}{\alpha} = v\left[1 - \frac{(\rho-\sigma)^2}{8(p^2+q^2)} - \frac{q(\rho+\sigma)(\rho-\sigma)^2}{8(p^2+q^2)^2} - \ldots\right], \quad . \quad (13)$$

and the *attenuation*

$$= \frac{q\alpha}{p} - \beta = \frac{1}{v}\left[\frac{\rho+\sigma}{2} + \frac{q(\rho-\sigma)^2}{4(p^2+q^2)} + \frac{(3q^2-p^2)(\rho+\sigma)(\rho-\sigma)^2}{16(p^2+q^2)^2} + \right]. \quad (14)$$

If $\rho=\sigma$, there is no distortion, the velocity of propagation is v, and the attenuation is $\frac{\mathrm{R}}{\mathrm{L}v}$ or $\frac{\mathrm{K}}{\mathrm{S}v}$ for all frequencies; and the damping has no effect on these quantities.

We have an interesting particular case when $\rho=\sigma=q$. Then $\beta=0$, and the state of affairs is given by

$$\mathrm{V} = \mathrm{V}_0 e^{-qt} \sin\left(pt - \frac{z}{v}\right).$$

Here the damping and the attenuation are balanced, so that the wave-train in the wires is at any instant *purely* simple harmonic throughout.

Numerical Values.—To obtain an estimate of the importance of the small terms of (13) and (14) I shall take the numbers given by Dr. Barton in his last paper (*loc. cit.*). Judging from his diagram (fig. 2) in that paper, the amplitude of the second positive maximum of the wave-train is about half that of the first. This would give

$$e^{\frac{2\pi q}{p}} = 2 \quad \text{or} \quad q = \frac{p \log_e 2}{2\pi}.$$

We have roughly

$$p=2\pi\times 35\times 10^6=22\times 10^7, \qquad q=24\times 10^6,$$
$$R=69{\cdot}5\times 10^5, \qquad L=19;$$
$$\therefore\quad \rho=37\times 10^4, \quad \text{and} \quad \sigma=0.$$

These values give for the velocity of propagation

$$v\{1-{\cdot}00000035\},$$

and for the attenuation

$$\frac{R}{2Lv}\{1+{\cdot}000091\},$$

so that the corrections are quite negligible. We see from the expressions (13), (14) that the damping q only affects the value of the small terms introduced by the inequality of ρ and σ.

Effect of a Terminal Resistance.—To find the effect on the incident waves of a resistance (without inductance) inserted between the ends of the wires, we put in the complex values in the expression (7) for the reflexion-factor. We then get

$$\frac{V_2}{V_1}=\frac{(\rho-q+v\beta x)+i(p-v\alpha x)}{(-\rho+q+v\beta x)-i(p+v\alpha x)}$$
$$=f+ig, \text{ say.}$$

Therefore to an incident wave e^{ipt} corresponds a reflected wave $(f+ig)e^{ipt}$; or, taking real parts, with incident $\cos pt$ we have reflected

$$f\cos pt-g\sin pt=\sqrt{f^2+g^2}\cos(pt+\theta),$$

where

$$\tan\theta=\frac{g}{f};$$

so that the change of amplitude is accompanied by a change of phase.

The values come out

$$f^2+g^2=\frac{(\rho-q+v\beta x)^2+(p-v\alpha x)^2}{(-\rho+q+v\beta x)^2+(p+v\alpha x)^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$=\frac{x^2v^2(\alpha^2+\beta^2)-2xv\{\beta(q-\rho)+\alpha p\}+(q-\rho)^2+p^2}{x^2v^2(\alpha^2+\beta^2)+2xv\{\beta(q-\rho)+\alpha p\}+(q-\rho)^2+p^2}, \quad (16)$$

$$\frac{g}{f}=\frac{2xv\{\beta p-\alpha(q-\rho)\}}{x^2v^2(\alpha^2+\beta^2)-(q-\rho)^2-p^2}. \qquad (17)$$

In order that there should be complete absorption of the incident waves it is necessary that the two squares in the numerator of (15) should vanish separately. This requires

$$\frac{v\alpha}{p}=\frac{v\beta}{q-\rho}=\lambda, \text{ say.}$$

If we substitute $v\alpha=p\lambda$ and $v\beta=(q-\rho)\lambda$ in equations (8) and (9), and eliminate λ by division, we find the condition reduces to

$$(\rho-\sigma)\{(q-\rho)^2+p^2\}=0, \qquad \therefore\ \rho=\sigma.$$

Therefore complete absorption is only attainable in the distortionless case. In general we can only reduce the reflected amplitude to a minimum.

We can write (16) and (17) in the forms

$$f^2+g^2=\frac{ax^2-2hx+b}{ax^2+2hx+b}, \qquad (18)$$

$$\tan\theta=\frac{g}{f}=\frac{2x\sqrt{ab-h^2}}{ax^2-b}. \qquad (19)$$

From (18) we see that to any value of reflected amplitude correspond two values of the terminal resistance, say x_1, x_2. We can show that the corresponding phase-differences θ_1, θ_2 are supplementary.

For from (18) we have

$$x_1x_2=\frac{b}{a};$$

$$\therefore\ \tan\theta_1+\tan\theta_2=2\sqrt{ab-h^2}\left[\frac{x_1}{ax_1^2-b}+\frac{x_2}{ax_2^2-b}\right]$$

$$=2\sqrt{ab-h^2}\,\frac{(x_1+x_2)(ax_1x_2-b)}{(ax_1^2-b)(ax_2^2-b)}=0.$$

The minimum reflected amplitude is got when

$$x_1=x_2=\sqrt{\frac{b}{a}}.$$

The reflexion-factor is then $\frac{\sqrt{ab}-h}{\sqrt{ab}+h}$, and the phase-difference is $\frac{\pi}{2}$. When $x=0$ we have complete reflexion with unaltered phase; with $x=\infty$, or the circuit open, we have complete reflexion with reversed phase. The simultaneous alteration of amplitude and phase-difference brings it about that we appear to pass continuously from amplitude $+1$ $(x=0)$ to amplitude -1 $(x=\infty)$ without passing through amplitude zero. This apparent anomaly was pointed out to me by Dr. Barton. Putting in the values of a, b, h, and substituting for $v\alpha$ and $v\beta$ approximate values from (11), (12), we find that the minimum value of the reflexion-factor $\sqrt{f^2+g^2}$ is

$$\frac{p(\rho-\sigma)}{4(p^2+q^2)},$$

and that the corresponding value of x is

$$1-\frac{q(\rho-\sigma)}{2(p^2+q^2)},$$

neglecting higher terms in $\rho\sigma$.

Numerical Values.—Again using Dr. Barton's numbers we get for the minimum reflexion-factor the value ·0004, and for the corresponding terminal resistance $Lv(1+·00009)$. If, therefore, the terminal resistance be adjusted until the reflected wave is a minimum, we may, without sensible error, take this resistance to be Lv, and ignore the reflected train altogether.

Queen's College, Belfast,
13th October, 1898.

Discussion.

Mr. Oliver Heaviside (communicated).—The connexion between the case investigated by Mr. Morton of a wave-train arising from a damped source and the standard case of an undamped source may be concisely exhibited thus. Using the notation of 'Electromagnetic Theory,' vol. i. p. 452, we have

$$V=\epsilon^{-qx}V_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$q=(R+Lp)^{\frac{1}{2}}(K+Sp)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

to express the wave-train V due to V_0 at $x=0$. When V_0 is simply periodic, say $=e \sin nt$, then $p=ni$ reduces q to $P+Qi$, given by

$$P \text{ or } Q=(\tfrac{1}{2})^{\frac{1}{2}}\{(R^2+L^2n^2)^{\frac{1}{2}}(K^2+S^2n^2)^{\frac{1}{2}}\pm(RK-LSn^2)\}^{\frac{1}{2}},$$

$$=(\tfrac{1}{2})^{\frac{1}{2}}\frac{n}{v}\left\{\left(1+\frac{R^2}{L^2n^2}\right)^{\frac{1}{2}}\left(1+\frac{K^2}{S^2n^2}\right)^{\frac{1}{2}}\pm\left(\frac{RK}{LSn^2}-1\right)\right\}^{\frac{1}{2}}, \quad (3)$$

so that the solution is

$$V=e\epsilon^{-Px}\sin(nt-Qx). \quad . \quad . \quad . \quad . \quad (4)$$

Now if V_0 be damped, say $=e_0\epsilon^{-at}\sin nt$, the effect of shifting ϵ^{-at} to the left is to change p to $p-a$ in the operator ϵ^{-qx}, that is, in q. This is the same as changing R to R_0 and K to K_0, given by

$$R_0=R-La, \qquad K_0=K-La, \quad . \quad . \quad . \quad . \quad (5)$$

so that the wave-train is

$$V=e_0\,\epsilon^{-at}\,\epsilon^{-P'x}\sin(nt-Q'x), \quad . \quad . \quad . \quad . \quad (6)$$

where P′ and Q′ are the same as P and Q with R_0 and K_0 instead of R and K. Of course R and K are positive, and q is in the first quadrant, but the new R_0 and K_0 may be positive or negative, and q may be in the second quadrant.

Practically, under the circumstances of the experiments,

$$Q'=\frac{n}{v} \quad \text{and} \quad P'=\frac{1}{2v}\left(\frac{R}{L}+\frac{K}{S}\right)-\frac{a}{v}, \quad . \quad . \quad . \quad (7)$$

where K is negligible, and

$$V=e_0\,\epsilon^{-at}\,\epsilon^{ax/v}\,\epsilon^{-Rx/2Lv}\sin(nt-nx/v). \quad . \quad . \quad . \quad (8)$$

As regards the cause of the attenuating coefficient $R/2Lv$ coming out by Dr. Barton's calculations from his experiments twice as great as when R is calculated by Lord Rayleigh's formula, I think it must be because the real circumstances do not correspond closely enough to those in the ideal theory. The external resistance, of unknown amount, is ignored, for one thing. Then again, it is not to be certainly expected that the formula in question is true for millions of vibrations per second. We can conclude from the experiments, though, that it furnishes an approximation to the real resistance.

But, even if it were rigorously true, the circumstances implied in it are not those in the experiments. The magnetic vibrations to which the wires are subjected are not long continued and undamped, as assumed in the formula. When a wave-train passes any point on a wire, its surface is subjected to an impulsive vibration lasting only a very minute fraction of a second, a vibration, moreover, which is very rapidly damped. So there is no definite resistance, and the resistance is greater than according to Lord Rayleigh's formula.

Perhaps, also, the terminal reflexions involved in Dr. Barton's calculations may introduce error.

XXXI. *Properties of Liquid Mixtures.*—Part III.* *Partially Miscible Liquids.* *By* R. A. LEHFELDT, *D.Sc.*†

THE phenomena of complete mixture between two liquids, about which so little systematic knowledge is yet in existence, are connected with the phenomena of ordinary solution by an intermediate stage, that in which two liquids dissolve one another to a limited extent only. The study of such couples seems a promising field of investigation, on account of the intermediate position they occupy; it seems to offer the chance of extending some of the laws arrived at with regard to simple solution to the more complicated cases. I have therefore attempted to get some information on the equilibrium between incomplete mixtures and the vapour over them, and especially at the "critical point," *i. e.*, the point at which incomplete miscibility passes over into complete. A recent short paper by Ostwald ‡ draws attention to the importance of that point in the theory of mixtures.

Choice of Liquids.

The first point is to obtain suitable pairs of liquids for experiment. In order to study the properties of the critical

* Part I. Phil. Mag. (5) xl. p. 398; Part II. Phil. Mag. (5) xlvi. p. 46; reprinted, Proc. Phys. Soc. xvi. p. 83.

† Read November 25, 1898.

‡ Wied. *Ann.* lxiii. p. 336 (1897).

point with ordinary vapour-pressure apparatus, it is necessary that the pressure at the critical point should be below one atmosphere, and that limits very much the choice of liquids. As a rule, two liquids either mix completely cold, or if they do not do that, raising to the boiling-point does not suffice to make them mix; two or three cases are all that I have been able to find in which the point of complete mixture can be arrived at by boiling, and consequently corresponds to a vapour-pressure below the atmospheric. Many other pairs of incompletely miscible liquids have been studied by Alexejew and others, but to arrive at their critical points it was necessary to raise them to a high temperature in sealed tubes. A recent paper by Victor Rothmund * contains new observations on the relation between concentration and temperature, including the concentration and temperature of the critical point, made by Alexejew's method. That paper contains a long account of previous work on the subject, which makes it the less necessary for me to go over the same ground. I will therefore mention only what has been done on vapour-pressures, as Rothmund does not touch on that side of the subject, merely adding two remarks to his paper. First, it does not seem to have been noticed that normal organic liquids always mix completely: I hoped to have found a normal pair to study first, in order to avoid the complication due to the abnormality supposed to be molecular aggregation in the liquid; I have not succeeded in finding such a pair. All the incompletely miscible pairs of liquids so far noted include water, methyl alcohol, or a low fatty acid as one member. To those with accessible critical points mentioned by Rothmund, I have only one pair to add, viz., ethylene dibromide and formic acid; these mix on boiling and separate into two layers when cold. I have not yet gone further with this couple; the vapour-pressure observations below refer to the well-known cases of phenol and water, aniline and water.

An account of previous experiments on the vapour-pressure of incompletely miscible liquids will indeed not take up much space, since, so far as I know, there is only one to record,

* *Zeitschr. f. phys. Chem.* xxvi. p. 433 (1898).

viz., Konowalow's * measurements on isobutyl alcohol-water mixtures. His observations (made by a static method) give some points on the vapour-pressure curve up to 100° for (1) pure isobutyl alcohol (100 %) ; (2) mixtures containing 94·05 % and 6·17 %, both clear; (3) an undetermined mixture which separated into two layers. He unfortunately did not measure the solubility of the alcohol in water, or water in the alcohol, at any of the temperatures for which vapour-pressure observations are recorded, so those data have to be supplied from Alexejew's results †. Konowalow, in the second part ‡ of his paper, proceeds to show that the possible forms of curve showing vapour-pressure against concentration (temperature constant) are two: (i.) the flat part of the curve bounded by a rising portion at one end and a falling portion at the other ; (ii.) the flat part bounded by a falling portion at each end. Isobutyl alcohol-water mixtures give a curve of the latter kind.

Isobutyl alcohol and water, however, possess a critical point at about 130°, *i. e.*, much above the boiling-point of either. I therefore decided to study first mixtures of phenol and water, which become homogeneous in any proportions below 70°.

The phenol was a commercial "pure" specimen ; to purify it further, it was placed in a distillation-flask and melted; then air was drawn through it for about half an hour, whilst its temperature was kept at about 160° to 170°, in order to dry it. It was then distilled, and by far the larger part came over between 178° and 180°. The fraction collected between 179°·5 and 180° (about half the mass) was used in the experiments. To make up mixtures, the process always adopted was to warm the stoppered bottle containing the phenol to just above the melting-point, and pour the required amount into a weighing-flask. It was found that the moisture absorbed from the air during the process was quite inappreciable. The phenol, kept day after day at 40° to 50° ready for use, slowly turned pink, showing the presence of rosolic acid ; but a comparative colour-observation showed that the

* Wied. *Ann.* xiv. p. 43.

† Wied. *Ann.* xxviii. p. 315.

‡ Wied. *Ann.* xiv. p. 222.

amount of impurity was probably not more than 1/10,000. When it was necessary to estimate phenol in a mixture, that was done by the method of Koppeschaar, tribromphenol being formed and the excess of bromine replaced by iodine and titrated with thiosulphate. The method gave quite satisfactory results.

Experimental Methods used.

The measurements on phenol-mixtures gave results contrary to my expectations, so that I became suspicious of the experimental methods. In the end I made use of four different kinds of apparatus, but found that they gave results in practical agreement, so that it became chiefly a question of convenience to decide between them.

The first method tried was the "dynamic," carried out with the same apparatus as described in Part II. It required no modification, except the use of a new thermometer, since the old one did not go above 60°. The new thermometer was a longer one, graduated in $\frac{1}{5}$ from 0° to 100° (by C. E. Müller, No. 8). Its corrections were obtained in two ways: first by comparison with a standard (Reichsanstalt, 7347) at certain fixed temperatures, viz., the boiling-points of methyl acetate (57°), methyl alcohol (65°), and ethyl alcohol (78°); secondly, by measurements of the vapour-pressures of water under the same conditions as in the actual experiments; in these conditions part of the stem was exposed.

To use the apparatus the required mixture was weighed out from melted phenol and distilled water, then warmed up in the weighing-bottle until it became homogeneous, and poured into the tube of the vapour-pressure apparatus. The apparatus works satisfactorily except for mixtures on a very steep part of the curve of vapour-pressure (p) over concentration (z); when dp/dz is great, the change of composition of the liquid, due to the evaporation, becomes disproportionately important, and the static method is to be preferred; in the case of phenol mixtures, however, that only affects a small part of the range —mixtures with 90 per cent. or more of phenol.

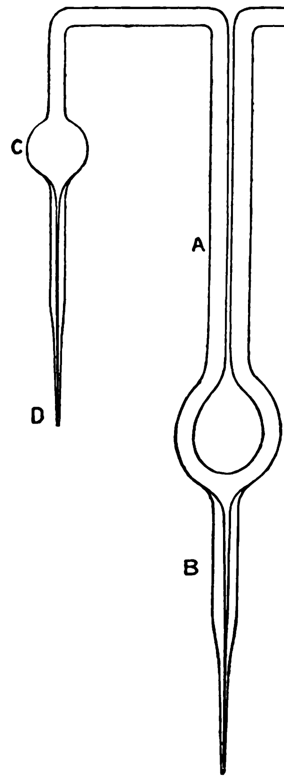
The anomalous result that made me at first doubtful of the accuracy of the method was that the addition of up to 60 or

70 per cent. of phenol to water made practically no difference to the vapour-pressure of the water. To check this, I made one or two experiments by the static method, in a barometer-tube surrounded by an alcohol-vapour jacket of the usual pattern. They were not carried out with any attempt at accuracy, but sufficed to show that the previous observations could not be far wrong. The problem then was to determine the small difference in pressure between water and the phenol-water mixtures, and as for that purpose a differential gauge is obviously more appropriate, I set about designing and making the apparatus described below. Its design is based on a point of technique that does not seem to be much known, and to which, therefore, I should like to draw attention. If a glass tube be drawn out fine, sealed at one end, and evacuated, the sealed end may be broken under the surface of a liquid, which then flows in at any desired rate according to the diameter of the tube, and the tube may at any moment be fused off in the middle by a mouth blowpipe, without any inconvenience whatever. This process of filling with a liquid will I think be found advantageous in many cases. The only trouble about it is to get the capillary of the right bore; since the rate of flow depends on the fourth power of the radius it is easy to make the tube too wide or too narrow. Of course I made a good many failures at first, but after some practice could rely on getting the required condition. I used tubing of about 1 millim. internal diameter, and 4 millim. external, and drew it out till the internal diameter was about one sixth of a millimetre; a few centimetres of such a bore gives a convenient rate of flow for liquids of the viscosity of water.

The apparatus for vapour-pressure measurements is shown in fig. 1. It consists of a U-tube, A, to serve as a gauge, carrying a branch, B, below, drawn out for filling as mentioned above. The top of the gauge-tube is bent round each side to the bulbs C, C′, which are also provided with filling-tubes D, D′. The whole is shown flat in the diagram; but as a matter of fact the side tubes C D and C′ D′ were bent round till the bulbs nearly touched, to ensure their being of the same temperature. The apparatus was cleaned out with chromic

acid, washed, and dried; the capillaries were then drawn out and two of them sealed up, the third being left with the bit of wider tubing beyond the capillary untouched. By means of this it was attached to a mercury-pump, exhausted, and the capillary fused. The point B was then opened under

Fig. 1.



mercury and fused off when the gauge contained sufficient: in the same way one of the bulbs was half filled with the mixture through D, and then the other with water (which must, of course, be freed from air) through D′. The apparatus, all of glass and hermetically sealed, is then ready for use: a glass millimetre scale is fastened with rubber bands to the gauge-tube, and the whole immersed in a large glass jar of

water. The scale was usually read by the telescope of a cathetometer, and sometimes the screw micrometer of the telescope was used to subdivide the graduation. The differential method avoids the necessity for any very great care in maintaining or measuring the temperature of the apparatus. It was found quite sufficient to heat the water-bath by leading a current of steam into it, and when the required temperature was reached, stop the steam for a moment and read the difference of level. When the highest temperature (90°) was reached, some of the water was siphoned off, replaced by cold, the whole mass well stirred, and a reading taken. There was no noticeable lag in the indications of the gauge, the readings at the same temperature, rising and falling, being in good agreement.

The fourth apparatus used was the Beckmann boiling-point apparatus, in its usual (second) form: with that observations at 100° were obtained of a kind to confirm the measurements made at somewhat lower temperatures with the vapour-pressure apparatus.

Observations of Vapour-pressure.

The following observations were obtained with the differential pressure-gauge: t is the temperature Centigrade, p the vapour-pressure of the mixture, π that of water; $\pi-p$ is therefore the difference observed with the gauge, and $(\pi-p)/\pi$ represents the relative lowering of the vapour-pressure of water by the addition of the quantity of phenol mentioned.

Phenol-water Mixtures.

67·36 per cent. of phenol.

t	50°	65°	75°	85°	90°
$\pi-p$	0·5	0·4	2·2	5·6	8·2
$(\pi-p)/\pi$......	0·005	0·002	0·008	0·013	0·015

77·82 per cent. of phenol.

t	70°	75°	80°	85°	90°
$\pi-p$	2·6	5·0	8·2	12·5	17·1
$(\pi-p)/\pi$......	0·011	0·017	0·023	0·029	0·032

82·70 per cent. of phenol.

t	40°	50°	60°	65°	70°	75°	80°	85°	90°
$\pi - p$	1·4	5·6	11·1	14·8	19·2	25·2	32·2	42·3	53·9
$(\pi - p)/\pi$..	0·025	0·061	0·075	0·079	0·082	0·087	0·091	0·098	0·103

90·46 per cent. of phenol.

t	25°	40°	50°	60°	65°	70°	75°
$\pi - p$	1·9	9·1	17·5	31·7	43·2	53·5	68·6
$(\pi - p)/\pi$...	0·081	0·165	0·190	0·213	0·231	0·230	0·237

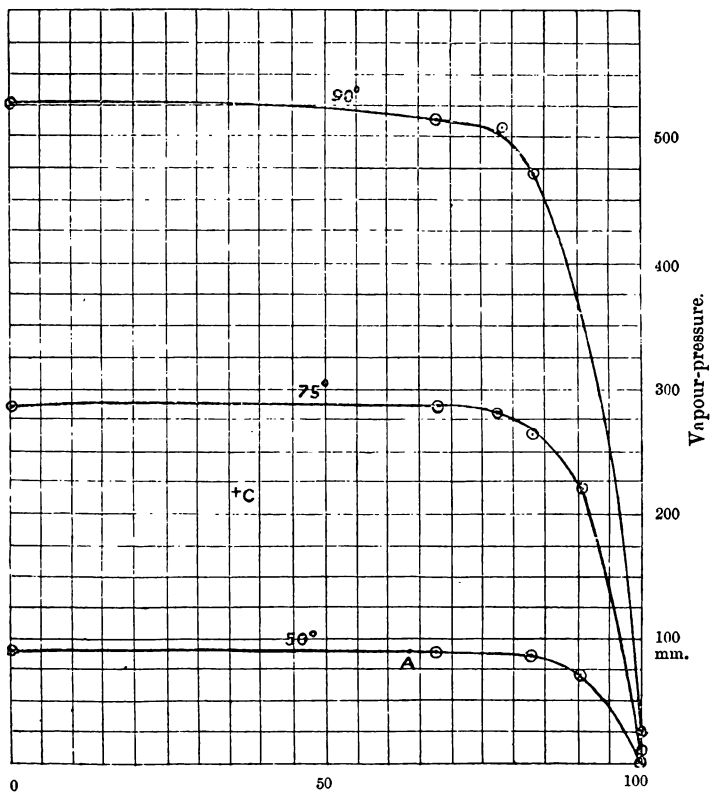
A mixture containing 7·74 per cent. of phenol gave no certain indication of a difference of pressure between the mixture and pure water. On this point, however, more reliable information is to be obtained with the Beckmann apparatus. It will be noticed from the preceding figures that the influence of the dissolved phenol becomes steadily greater as the temperature rises, *e.g.* 82 per cent. of phenol produces nearly twice as great a change of vapour-pressure proportionally at 90° as it does at 50°. In agreement with this the rise of vapour-pressures in dilute solutions of phenol is more marked at 100° than at the lower temperatures at which the vapour-pressure apparatus is available. The result of an experiment on the boiling-point is as follows:—

Per cent. phenol.	Fall of boiling-point.	Corresponding rise of vapour-pressure, $p - \pi$.
4·8	0·154	4·1
9·0	0·169	4·5
13·0	0·161	4·3
16·4	0·154	4·1

The general character of the results is sufficiently shown by fig. 2, in which the isothermals of 90° and 75° are sufficiently represented. That of 90° is comparable with the curve for alcohol-toluene mixtures (see fig. 2 in the preceding memoir), only that the flatness extending over a great part of the range of concentrations is exaggerated in the phenol-water mixtures. The curve for 75°—still above the critical point—

is still flatter; indeed it is imposible to say whether it rises or falls. Probably, therefore, below the critical point (where the vapour-pressure of phenol is inconveniently small for measurement) the isothermal, instead of consisting of a horizontal line bounded by two curves, would consist of a horizontal passing through the point representing pure water,

Fig. 2.—Isothermals of Phenol-Water Mixtures.



bounded at the other end only by a descending curve. Such an isothermal—that of 50°—is shown in fig. 2. The horizontal part ends at the point A (63 per cent.), beyond which the mixtures are homogeneous: the curve beyond A may theoretically meet the horizontal line at a finite angle, but

that is certainly not distinguishable on the diagram. The curve is in fact exactly similar, so far as the experiments show, to that for 75°, and the pressure of the critical point, C on the diagram, which lies between them (at 68°·4) appears to make no difference whatever in this case—a case of great disparity in the vapour-pressures of the two components.

For comparison, a few experiments were made with aniline (not specially purified) and water. A mixture which consisted of two layers, even at the highest temperature used in the experiment, gave in the differential apparatus the following results :—

t	70°	80°	85°
$\pi - p$	−6·8	−11·1	−13·3
$(\pi - p)/\pi$ ...	−0·0290	−0·0315	−0·0307

Whilst the Beckmann apparatus gave at 100° :—

Per cent. aniline.	Lowering of boiling-point.	$\pi - p$.	$\frac{\pi - p}{\pi}$.
3·99	0·636	−17·1	0·0225
7·68	0·921	−24·6	0·0324
11·10	0·921	−24·6	0·0324

The second column gives the observed fall in temperature on adding aniline to the water; the third column the rise of pressure corresponding, at the rate of 26·8 millim. per degree. Water at 100° is saturated by the addition of 6·5 per cent. of aniline, and it will be seen that the vapour-pressure rises no further after that. The relative rise of vapour-pressure on saturation is 0·0324 at 100°, in satisfactory agreement with the numbers obtained by the differential gauge (0·0290, 0·0315, 0·0307), a tendency to increase with temperature being distinguishable here, as with phenol. Now suppose the vapour-pressure of a saturated mixture to be obtained in this way: let the partial pressure of the water-vapour be that of pure water reduced by the normal amount (Raoult's law) due to the solution in it of the maximum quantity of aniline: and let the partial pressure of aniline-vapour be that

of pure aniline reduced by the normal amount due to the solution in it of the maximum quantity of water. We get the following results at 100° :—

Vapour-pressure of water = 760 millim.
Solubility of aniline = 6·5 per cent. = 1·32 molecular per cent.
Partial pressure of water = 98·68 per cent. of 760 = 749·9 millim.
Vapour-pressure of aniline = 46 millim.*
Solubility of water in aniline = 8·7 per cent. = 33 molecular per cent. (Alexejew).
Partial pressure of aniline = 30·8.
Total pressure = 749·9 + 30·8 = 780·7.
Observed pressure = 784·6 millim.

The vapour-pressure of the saturated mixture is therefore given fairly well by the above rule. The rule cannot be applied to phenol mixtures, as below the critical point the vapour-pressure of phenol is too low to determine with accuracy. Konowalow's measurements of the vapour-pressure of isobutyl alcohol-water mixtures, combined with Alexejew's measurements of solubility, give the following results. At 90° :—

Vapour-pressure of water = 525 millim.
Solubility of isobutyl alcohol = 7 per cent. = 1·8 molecular per cent.
Partial pressure of water-vapour = 98·2 per cent. of 525 = 515·5.
Vapour-pressure of isobutyl alcohol = 378 millim.
Solubility of water = 25 per cent. = 57·8 molecular per cent.
Partial pressure of isobutyl alcohol = 159·5.
Sum = 675·1.
Observed pressure = 767.

In this case the alcohol saturated with water contains more molecules of water than of alcohol, and it is not to be expected that the normal depression of the vapour-pressure should

* Kahlbaum, *Zeitsch. f. phys. Chem.* xxvi. p. 604.

2 A 2

hold over so wide a range as 57·8 per cent. The numbers in fact show that the partial pressure of isobutyl alcohol must be very much greater—about 250 millim. The curve of partial pressures is therefore comparable with that for ethyl alcohol in benzene and toluene (see Part II. tables pp. 95–96).

Characteristic Surface for Phenol-Water Mixtures.

To complete an account of the behaviour of phenol-water mixtures, it is necessary to draw a diagram of the relations between temperature and concentration ; this is given in

Fig. 3.

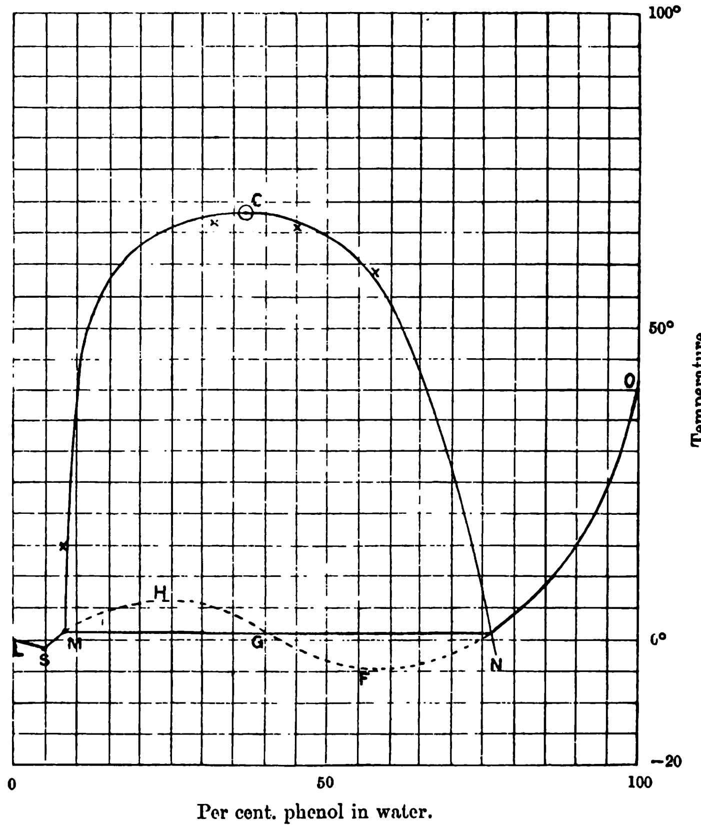


fig. 3. Figs. 2 and 3 together, therefore, give a notion of the shape of the "characteristic surface," *i. e.* the surface showing

the relations between concentration, temperature, and pressure. Fig. 2 contains three sections at right angles to the axis of temperature (for $T=90°$, $T=75°$, $T=50°$ respectively), while fig. 3 gives one section at right angles to the axis of pressure ($p=1$ atmo).

The behaviour of phenol-water mixtures is formally identical with that of benzoic acid and water *, but the curve branches of the diagram are of very different relative sizes to those of the last-named mixtures. The features of the diagram are as follows :—

L. Freezing-point of water.
O. Freezing-point of phenol.
LS. Freezing-point of aqueous solutions of phenol.
ONFGHMS. Freezing-point of solutions of water in phenol.
S. Cryohydric point.
MC. Saturation of water with phenol.
NC. Saturation of (liquid) phenol with water.
C. Critical point of mixture.

The line LS is given by the thermodynamic equation

$$t=\frac{0\cdot02T^2}{L}$$

where T is the absolute temperature of fusion of ice, L the latent heat of fusion, and t the resulting molecular depression of the freezing-point ; it accordingly starts with a slope of $0°\cdot2$ for one per cent. of phenol. The initial slope of ON is given by a similar equation, and is $4°\cdot15$ for one per cent. of water; a direct observation gave as a point of the curve

80·5 per cent. phenol,
melting-point + 5·0.

This is marked with a dot in the figure, and lies to the right of CN; by continuing the curve through the point so found until CN is met, we reach the point N where the phenol is saturated with water ; on increasing the concentration a second liquid layer appears, consisting of water saturated with phenol. NFGHM is purely hypothetical, referring to unstable mixtures ; actually any mixture of concentration

* See van't Hoff, *Vorlesungen über theoretische und physikalische Chemie*, Heft i. p. 48. (Braunschweig, 1898.)

between 8 per cent. and 77 per cent. of phenol will separate into two layers on cooling, and on further reduction of temperature freeze at the constant temperature (about $+1^\circ\cdot5$) represented by the horizontal straight line MN. The cryohydric point lies to the left of the saturation curve CM, so that it is actually attainable: its existence was shown by making a solution containing 5·25 per cent. of phenol, and cooling it in a bath of ice and salt: it began to freeze without previous separation into two layers, and the temperature remained constant at $-0^\circ\cdot9$. About half of it was frozen, the beaker removed from the freezing-mixture, and some of the liquid remaining poured off for analysis; it was found to contain 4·83 per cent. phenol: this concentration is therefore in equilibrium with both ice and solid phenol which had been deposited on the sides of the beaker. The cryohydric mixture therefore contains so little phenol that it may be looked upon as a dilute solution of phenol in water, and its calculated freezing-point, according to van't Hoff's rule, would be $-1^\circ\cdot0$, in agreement with the observed value $-0^\circ\cdot9$. Consequently solutions of strength between M and S will deposit phenol on cooling, those between L and S (0 to 4·83 per cent.) ice.

The diagram is completed by the curve MCN which is drawn from Rothmund's observations*, which are indicated by dots; my own observations (shown by crosses) are in practical agreement with his and Alexejew's.

Finally, the curves divide the diagram into regions, with the following meanings:—

Below LS undercooled solutions of phenol, from which ice crystallizes out, with formation of the saturated solutions LS.

Below SMNO supersaturated solutions of phenol, from which phenol crystallizes out with formation of the saturated solutions of phenol in water (SM) and water in phenol (NO)

MCNGM, unstable mixtures which separate into the two saturated solutions CM and CN, forming two liquid layers.

Above LSMCNO homogeneous liquid mixtures.

The Davy-Faraday Research Laboratory, Royal Institution,
London, October, 1898.

* *L. c.* p. 452.

Discussion.

Prof. S. Young (communicated).—The fact pointed out by Mr. Lehfeldt that normal organic liquids always mix completely is a curious one. I should be inclined myself to qualify the statement by adding the words—"so far as our present knowlege goes."

At any rate there are certainly pairs of normal organic liquids which, though miscible in all proportions, approximate closely in their behaviour to partially miscible liquids.

Such a pair is benzene (B.P. 80°) and normal hexane (B.P. 69°).

When American petroleum is fractionally distilled it is found that the benzene, which is present in small quantity, does not come over at about 80° but mostly at about 65°, and the most probable explanation appeared to me to be that benzene and hexane behave, as regards distillation, like partially miscible liquids. This view has been fully confirmed by an investigation of the boiling-points and also the specific gravities of mixtures of the two hydrocarbons, an account of which has just been read before the Chemical Society by Dr. D. H. Jackson and myself.

The boiling-point curve is similar in general form to that of phenol and water, as shown by Mr. Lehfeldt, though the deviation from the ordinary form is not so marked.

10 % of benzene has practically no influence on the B.P. of normal hexane, but 10 % of hexane lowers the B.P. of benzene nearly 3°. Also there is always expansion on mixing benzene and hexane, the maximum reaching about 0·4 per cent.

Prof. S. P. Thompson asked whether any relation had been observed between the vapour-pressure and the surface-tension of the mixtures.

Dr. Lehfeldt, in reply, said he was not sure whether the surface-tensions of the components pass into one another at the critical point of mixture.

XXXII. *Longitudinal Vibrations in Solid and Hollow Cylinders.* By C. Chree, *Sc.D., LL.D., F.R.S.**

Preliminary.

§ 1. The frequency $k/2\pi$ of longitudinal vibrations in the ideal isotropic bar of infinitely small cross section has long been known to be given by

$$k=p\sqrt{\mathrm{E}/\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ρ is the density, E Young's modulus ; p is given by

$p=i\pi/l$ when the ends are both free or both fixed,

$p=(2i+1)\pi/(2l)$ when one end is free, the other fixed,

l being the length of the rod and i a positive integer.

For the fundamental or lowest note $i=1$.

For a circular bar whose radius a, though small compared to l, is not wholly negligible, the closer approximation

$$k=p(\mathrm{E}/\rho)^{\frac{1}{2}}\{1-\tfrac{1}{4}p^2\eta^2a^2\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where η is Poisson's ratio, was obtained independently by Prof. Pochhammer † and Lord Rayleigh‡ fully 20 years ago.

§ 2. The subject has been treated by myself in three papers in the 'Quarterly Journal of ... Mathematics' (A) (p. 287, 1886), (B) (p. 317, 1889), (C) (p. 340, 1890).

In (A) I arrived at (2) describing it (*l. c.* p. 296) as "obtained as a second approximation by Lord Rayleigh." I further said, "We do not think, however, that his proof affords any means of judging of the degree of accuracy of the result, as it is founded on a more or less probable hypothesis and does not profess to be rigid." I subsequently learned that Lord Rayleigh did not admit any want of rigidity in his proof, and it appears without modification in the second edition of his Treatise on 'Sound.' I much regret having to differ from so eminent an authority, but I have not altered my original opinion.

* Read December 9, 1898.

† *Crelle*, vol. lxxxi. (1876).

‡ 'Theory of Sound,' vol. i. art. 157.

In (B) I reached the more general result

$$k=p(\mathrm{E}/\rho)^{\frac{1}{2}}(1-\tfrac{1}{2}p^2\eta^2\kappa^2), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where κ is the radius of gyration of the cross section of the rod about its axis. This was established by a strict elastic solid method for an elliptic section, and in a somewhat less rigid way for a rectangular section. (A) and (B) were confined, like the investigations of Lord Rayleigh and Professor Pochhammer, to isotropic materials.

In (C) I considered the more general case of an æolotropic bar whose long axis was an axis of material symmetry, and found by strict elastic solid methods that (2) still held for a circular section, if E denoted Young's modulus for stress along the length of the bar, and η Poisson's ratio for the consequent perpendicular contractions. Further, applying Lord Rayleigh's method, modified in a way I deem necessary, I obtained (3) for any form of cross section.

§ 3. Since the publication of (C) Mr. Love has discussed the subject in vol. ii. of his 'Treatise on Elasticity.' On his p. 119 * he refers to (2) as "first given by Prof. Pochhammer... and afterwards apparently independently by Mr. Chree." Again, in the new edition of his 'Sound' Lord Rayleigh, after deducing (2), says "A more complete solution... has been given by Pochhammer... A similar investigation has also been published by Chree."

In view of these remarks, I take this opportunity of stating explicitly:—

1. That Pochhammer's work was wholly unknown to me until the appearance of Love's 'Elasticity.'

2. That my method of solution in (A) is essentially different from Pochhammer's, while the methods in (B) and (C) are absolutely different from his. The method of (A) agrees with Pochhammer's in employing the equations of elasticity in cylindrical coordinates. After obtaining, however,—as is customary in most elastic problems—the differential equation for the dilatation, Pochhammer obtains a differential equation for the quantity

$$\frac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dr}\right),$$

* The preface, p. 13, describes the result as "obtained independently" by me.

u and w being the displacements parallel to the radius r and the axis z, and uses this quantity as a stepping-stone to the values of u and w. On the other hand, I succeeded in separating u and w so as to obtain at once two differential equations, in one of which u appeared alone with the dilatation, while the other contained only w and the dilatation (see (28) and (29) later).

§ 4. There are two other points in Mr. Love's Treatise to which I should like to refer. In his art. 263 he substitutes the term *extensional* for *longitudinal*, adding in explanation, "The vibrations here considered are the 'longitudinal' vibrations of Lord Rayleigh's *Theory of Sound*. We have described them as 'extensional,' to avoid the suggestion that there is no lateral motion of the parts of the rod."

I am altogether in sympathy with the object which Mr. Love has in view (I expressed myself somewhat strongly on the point in (A) p. 296, and (C) pp. 351–2), but I doubt the wisdom of attempting to displace a term so generally adopted as *longitudinal*.

In the second matter I regret to find myself at variance with Mr. Love. Referring to transverse vibrations in a rod, he says on p. 124 of his vol. ii., "the boundary conditions at free ends cannot be satisfied exactly ... as they can in the ... extensional (longitudinal) modes." In reality, however, the boundary equations at a free end are not exactly satisfied in the case of longitudinal vibrations either by Pochhammer's solution or my own. The slip may be a purely verbal one on Mr. Love's part, but his readers might be led to accept the statement as accurate, owing to a slight error in the expression for the shearing stress $\widehat{zr}$ near the top of Mr. Love's p. 120. We find there

$$\widehat{zr}=2\mu\{\gamma A\frac{d}{dr}J_0(\kappa' r)+\dots\}e^{\iota(\gamma z+pt)},$$

where $\iota=\sqrt{-1}$.

The correct expression (compare Mr. Love's second boundary equation on p. 118) is

$$\widehat{zr}=\iota\mu\{2\gamma A\frac{d}{dr}J_0(\kappa' r)+\dots\}e^{\iota(\gamma z+pt)}.$$

Owing to the omission of ι in the expression on p. 120 it looks as if $\widehat{zr}$ vanished for the same values of z as the normal stress $\widehat{zz}$. In reality, as I showed in (A) p. 295, $\widehat{zr}$ does not vanish over a terminal free section of radius a, but is of the order $r(r^2-a^2)/l^3$, where r is the perpendicular on the axis. We are quite justified in neglecting $\widehat{zr}$ when terms of order $(a/l)^3$ are negligible, but strictly speaking the solution is so far only an approximate one when the ends are free.

A New Method.

§ 5. In the Camb. Phil. Soc. Trans. vol. xv. pp. 313–337 I showed how the mean values of the strains and stresses might be obtained in any elastic solid problem independently of a complete solution. For isotropic materials I obtained (*l. c.* p. 318) three formulæ of the type

$$\mathrm{E}\iiint\frac{d\gamma}{dz}dx\,dy\,dz=\iiint\{\mathrm{Z}z-\eta(\mathrm{X}x+\mathrm{Y}y)\}\,dx\,dy\,dz$$
$$+\iint\{\mathrm{H}z-\eta(\mathrm{F}x+\mathrm{G}y)\}\,d\mathrm{S}, \quad . \quad (4)$$

where α, β, γ are components of displacement, X, Y, Z of bodily forces, and F, G, H of surface forces. The volume integrals extend throughout the entire volume, and the surface integrals over the entire surface of the solid. As was explicitly stated in proving the results (*l. c.* p. 315), X, Y, Z may include 'reversed effective forces'

$$-\rho\frac{d^2\alpha}{dt^2},\quad -\rho\frac{d^2\beta}{dt^2},\quad -\rho\frac{d^2\gamma}{dt^2},$$

where ρ is the density.

In the present application there are no real bodily forces; we may also leave surface forces out of account, if we suppose that when one end of a rod is held, that end lies in the plane $z=0$.

Supposing the rod to vibrate with frequency $k/2\pi$, we have

$$\mathrm{X}=-\rho\frac{d^2\alpha}{dt^2}=k^2\rho\alpha,\ \ \&\text{c.},$$

so that we replace (4) by

$$E\iiint \frac{d\gamma}{dz}\,dx\,dy\,dz = k^2\rho\iiint \{\gamma z - \eta(\alpha x + \beta y)\}\,dx\,dy\,dz, \quad . \quad (5)$$

and similarly with the two other equations of the same type.

The only other result required is one established in my paper (B): viz., that the general solution of the elastic solid equations of motion in which the terms contain $\cos pz$ or $\sin pz$ consists of two independent parts. In the first, which alone applies to longitudinal vibrations, α and β are odd and γ an even function of x and y.

We may thus assume

$$\left.\begin{aligned}\alpha &= \cos kt \cos (pz-\epsilon)\{A_1x + A_1'y + A_3x^3 + A_3'x^2y \\ &\qquad + A_3''xy^2 + A_3'''y^3 + \ldots\}, \\ \beta &= \cos kt \cos (pz-\epsilon)\{B_1x + B_1'y + B_3x^3 + \ldots\}, \\ \gamma &= \cos kt \sin (pz-\epsilon)\{C_0 + C_2x^2 + C_2'xy + C_2''y^2 + \ldots\},\end{aligned}\right\} \quad . \quad (6)$$

where ϵ is a constant depending on the position of the origin of coordinates and the terminal conditions. It is obvious from various considerations that the same ϵ occurs in the values of α, β, γ. Certain relations must subsist between the constants A, B, C in the above expressions, in virtue of the body-stress equations, but we do not require to take any heed of these for our present purpose.

§ 6. As the validity of solutions in series has been a subject of contention in other elastic solid problems, some doubt may be entertained as to the results (6). I would be the last to deny the reasonableness of this, because I do not myself regard (6) as universally applicable.

According to my investigations, quantities such as (A_3x^3/A_1x) are of the order (greatest diameter/nodal length)^2, and the series become less rapidly convergent as (greatest diameter/nodal length) increases. In other words, increase either in (greatest diameter/rod length) or in the order of the "harmonic" of the fundamental note reduces the rapidity of convergence. The proper interpretation, however, to put on this is not that (6) is a wrong formula for *longitudinal* vibrations, but simply that under the conditions specified the vibrations tend to depart too widely from the longitudinal type. If we apply this solution the results deduced from it

themselves tend to show the degree of rapidity of the convergence, and what we have to do is to keep our eye on the results and accept them only so long as they are consistent with rapid convergency.

Perhaps the following *résumé* of my views on this point may be useful :—

1°. In obtaining (6) originally I employed the *complete* elastic solid equations for isotropic materials. In other cases where difficulties have arisen over expansion in series, they seem mainly due to the fact that the elastic solid equations have been whittled down in the first instance for purposes of simplicity. When one omits terms in a differential equation for diplomatic reasons, the results may be perfectly satisfactory under certain limitations. Owing, however, to the mutilation of the differential equations, the resulting solutions are unlikely to contain within themselves any satisfactory indication of the limits to their usefulness. It is very much a case of running a steam-engine without a safety-valve.

2°. When the bar is of circular section and isotropic, the series occurring in (6) are Bessel's functions of a well-known type, whose rapidity of convergence appears well ascertained under the normal conditions of the problem. When the section is circular, and the material not isotropic but symmetrical round the axis, the series, whose mathematical law of development I have obtained, converge to all appearance quite as rapidly as in the case of isotropy. The other cases of isotropic material—sections elliptical or rectangular—which I have considered present similar features ; the only difference being that the rate of convergence diminishes with increase in any one dimension of the cross section.

3°. If we suppose $k=0$, or the vibrations to be of infinite period, the solution must reduce to that for the equilibrium of a rod under uniform longitudinal traction. Now, in the case of equilibrium γ/z reduces to a constant, while α and β are linear in x and y for *all forms of cross section*. The commencing terms in series (6) are thus of the proper form under all conditions, and the form of the differential equations shows that if α, for instance, contains a term in x it must contain terms in x^3, xy^2, and other *integral* powers of x and y.

4°. The general type of the differential equations is the

same for all kinds of elastic material, isotropic or æolotropic, and the surface conditions are identical in all cases; thus the type of solution must always be the same. The results may become enormously lengthy for complicated kinds of æolotropy, but by putting a variety of the elastic constants equal to one another we must reduce the most complicated of these expressions to coincidence with the corresponding results for isotropy. Of course it does not follow that the convergence will be equally rapid for all materials. A large value of a Poisson's ratio in conjunction with an elongated dimension of the cross section may reduce the convergency so much as to throw the higher "harmonics" outside the pale of longitudinal vibrations.

5°. The more the section departs from the circular form the less rapid in general is the convergence, and the larger the correction supplied by the second approximation. In fact the size of the correction is probably the best criterion by which to judge of the limitations of our results. If the correction is large even for the fundamental note, it is pretty safe to conclude that the section is not one adapted for the ordinary type of longitudinal vibrations. If a section, for instance, were of an acutely stellate character, with a lot of rays absent and the centroid external to the material, I for one should be extremely chary of applying to it the ordinary formula.

§ 7. For definiteness let us consider the fixed-free vibrations, taking the origin of coordinates at the centroid of the fixed end. Our terminal conditions are

$$\gamma=0 \text{ when } z=0,$$

$$\widehat{zz}=0 \quad ,, \quad z=l;$$

the latter condition being the same thing as

$$\frac{d\gamma}{dz}=0 \text{ when } z=l.$$

These conditions give at once

$$\epsilon=0, \quad p=(2i+1)\pi/2l,$$

and hence

$$\int_0^l pz \sin(pz-\epsilon)dz=\int_0^l \cos(pz-\epsilon)dz.$$

Take the axes of x and y along the principal axes of the cross section σ, so that

$$\iint xy\,dx\,dy=0,$$

$$\iint x^2 dx\,dy=\kappa_2{}^2\sigma,\quad \iint y^2 dx\,dy=\kappa_1{}^2\sigma.$$

Then, substituting from (6) in (5), we obtain at once

$$(\mathrm{E}p-k^2\rho/p)(\mathrm{C}_0+\mathrm{C}_2\kappa_2{}^2+\mathrm{C}_2{}''\kappa_1{}^2+\ldots)$$
$$=-k^2\rho\eta(\mathrm{A}_1\kappa_2{}^2+\mathrm{B}_1{}'\kappa_1{}^2+\ldots)\quad . \quad . \quad (7)$$

The section is supposed small, *i. e.* terms in $\kappa_1{}^2$, $\kappa_2{}^2$ are small compared to C_0, though large compared to the terms of orders $\kappa_1{}^4$, &c., which are omitted in the above equation. Thus as a first approximation the coefficient of C_0 must vanish, or

$$k=p\sqrt{\mathrm{E}/\rho},\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is simply the ordinary frequency equation.

Treating the other two equations of the type (5) similarly, we obtain the two results

$$\mathrm{E}(\mathrm{A}_1+3\mathrm{A}_3\kappa_2{}^2+\mathrm{A}_3{}''\kappa_1{}^2+\ldots)$$
$$=-k^2\rho\left\{\frac{\eta}{p}(\mathrm{C}_0+\mathrm{C}_2\kappa_2{}^2+\mathrm{C}_2{}''\kappa_1{}^2+\ldots)+\eta\mathrm{B}_1{}'\kappa_1{}^2-\mathrm{A}_1\kappa_2{}^2+\ldots\right\},\ (8)$$

$$\mathrm{E}(\mathrm{B}_1{}'+\mathrm{B}_3{}'\kappa_2{}^2+3\mathrm{B}_3{}'''\kappa_1{}^2+\ldots)$$
$$=-k^2\rho\left\{\frac{\eta}{p}(\mathrm{C}_0+\mathrm{C}_2\kappa_2{}^2+\mathrm{C}_2{}''\kappa_1{}^2+\ldots)+\eta\mathrm{A}_1\kappa_2{}^2-\mathrm{B}_1{}'\kappa_1{}^2+\ldots\right\}.\ (9)$$

The terms not shown are of the order $\kappa_1{}^4$, $\kappa_2{}^4$, or higher powers of κ_1 and κ_2. Combining (8) and (9) we get

$$\mathrm{E}(\mathrm{A}_1-\mathrm{B}_1{}')=k^2\rho(1+\eta)(\mathrm{A}_1\kappa_2{}^2-\mathrm{B}_1{}'\kappa_1{}^2)+\ldots,\quad . \quad . \quad (10)$$

$$\mathrm{E}(\mathrm{A}_1+\mathrm{B}_1{}')=-2k^2\rho(\eta/p)(\mathrm{C}_0+\mathrm{C}_2\kappa_2{}^2+\mathrm{C}_2{}''\kappa_1{}^2+\ldots)$$
$$+(1-\eta)k^2\rho(\mathrm{A}_1\kappa_2{}^2+\mathrm{B}_1{}'\kappa_1{}^2)+\ldots\quad . \quad (11)$$

To see the significance of (10) replace $k^2\rho$ by its approximate value $p^2\mathrm{E}$, when we have

$$\mathrm{A}_1-\mathrm{B}_1{}'=(1+\eta)\{\mathrm{A}_1(p\kappa_2)^2-\mathrm{B}_1{}'(p\kappa_1)^2\}+\ldots\quad . \quad (12)$$

As we have seen, $p=(i+\frac{1}{2})\pi/l$; and thus, so long as i is not too large, $(p\kappa_2)^2$ and $(p\kappa_1)^2$ are in a thin rod small quantities

of the orders $(\kappa_2/l)^2$ and $(\kappa_1/l)^2$. Hence we deduce from (12) as a first approximation

$$B_1'=A_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This is all we require for our present purpose; but, in passing, it may be noted that as a second approximation we have

$$B_1'=A_1\{1+(1+\eta)p^2(\kappa_1^2-\kappa_2^2)\}.$$

The more the section departs from circularity—*i.e.* the more elongated it is in one direction—the greater is the difference between B_1' and A_1. This and the fact that $i\kappa_1/l$ and $i\kappa_2/l$ must both remain small are useful indications of the limitations implied in the method of solution.

Employing (13) in (7) and (11) we have, neglecting smaller terms,

$$(Ep-k^2\rho/p)(C_0+C_2\kappa_2^2+C_2''\kappa_1^2)=-k^2\rho\eta A_1(\kappa_2^2+\kappa_1^2),$$
$$2k^2\rho(\eta/p)(C_0+C_2\kappa_2^2+C_2''\kappa_1^2)=-2EA_1.$$

Whence we deduce at once, without knowing anything of the constants C_2, A_3, &c.,

$$(Ep-k^2\rho/p)\div(2k^2\rho\eta/p)=k^2\rho\eta(\kappa_2^2+\kappa_1^2)/(2E).$$

Using in the small term (that containing k^4) the first approximation (1), we have

$$k^2\rho=E\,p^2\{1-\eta^2p^2(\kappa_1^2+\kappa_2^2)\}; \quad . \quad . \quad . \quad (14)$$

and this, as

$$\kappa_1^2+\kappa_2^2=\kappa^2,$$

simply reduces to (3).

That the above proof is as satisfactory in every way as one based on ordinary elastic solid methods I should hesitate to maintain. Unless one knew beforehand a good deal about the problem there would, I fear, be considerable risk of misadventure.

§ 8. In illustrating the method in detail I have selected the case of isotropic material simply because I did not wish to frighten my readers. The assumption of isotropy almost invariably shortens the mathematical expressions, and generally also simplifies the character of the mathematical operations; and isotropic solids thus flourish in the text-books to a

much greater degree than they do in nature. When, however, the mathematical difficulties are trifling, as in the present case, it seems worth while considering some less specialized material. I shall thus briefly indicate the application of the method to the case of material symmetrical with respect to the three rectangular planes of x, y, z, taken as in the previous example. In this case the stress-strain relations involve, on the usual hypothesis, nine elastic constants. Such quantities as Young's modulus or Poisson's ratio must be defined by reference to directions. Thus let E_1, E_2, E_3 denote the three principal Young's moduli, the directions 1, 2, 3 being taken along the axes of x, y, z respectively. There are six corresponding Poisson's ratios, each being defined by two suffixes, the first indicating the direction of the longitudinal pull, the second that of the contraction. For instance, η_{12} applies in the case of the contraction parallel to the y-axis due to pull parallel to the x-axis. The order of the suffixes is not immaterial, but there exist the following relations :—

$$\eta_{12}/E_1=\eta_{21}/E_2\ ;\quad \eta_{23}/E_2=\eta_{32}/E_3\ ;\quad \eta_{31}/E_3=\eta_{13}/E_1. \quad (15)$$

The three equations answering to (4) are

$$E_3\iiint\frac{d\gamma}{dz}dz\,dy\,dz=k^2\rho\iiint\{\gamma z-\eta_{31}\alpha x-\eta_{32}\beta y\}\,dx\,dy\,dz, \quad (16)$$

$$E_1\iiint\frac{d\alpha}{dx}dx\,dy\,dz=k^2\rho\iiint\{\alpha x-\eta_{12}\beta y-\eta_{13}\gamma z\}\,dx\,dy\,dz, \quad (17)$$

$$E_2\iiint\frac{d\beta}{dy}\,dx\,dy\,dz=k^2\rho\iiint\{\beta y-\eta_{23}\gamma z-\eta_{21}\alpha x\}\,dx\,dy\,dz. \quad (18)$$

From the nature of the elastic solid equations the expressions for the displacements must be of the same general form as for isotropy, so that we may still apply the formulæ (6) for α, β, γ. Doing so, and following exactly the same procedure as in the case of isotropy, we obtain from (16), for any shape of section,

$$(E_3p-k^2\rho/p)(C_0+\ldots)=-k^2\rho(\eta_{31}A_1\kappa_2^{\,2}+\eta_{32}B_1'\kappa_1^{\,2})+\ldots, \quad (19)$$

and from (17) and (18) as first approximations

$$A_1/(\eta_{13}E_3/E_1)=B_1'/(\eta_{23}E_3/E_2)=-p(C_0+\ldots).\ . \quad (20)$$

Thence we obtain at once

$$k^2\rho = p^2E_3\left\{1 - p^2E_3\left(\frac{\eta_{31}\eta_{13}}{E_1}\kappa_2^2 + \frac{\eta_{32}\eta_{23}}{E_2}\kappa_1^2\right)\right\}. \qquad (21)$$

Employing (15) we give this the more elegant form

$$k^2\rho = p^2E_3\{1 - p^2(\eta_{31}{}^2\kappa_2{}^2 + \eta_{32}{}^2\kappa_1{}^2)\},$$

whence

$$k = p(E_3/\rho)^{\frac{1}{2}}\{1 - \tfrac{1}{2}p^2(\eta_{31}{}^2\kappa_2{}^2 + \eta_{32}{}^2\kappa_1{}^2)\}. \quad . \quad (22)$$

For a circular section of radius a

$$k = p(E_3/\rho)^{\frac{1}{2}}\{1 - \tfrac{1}{8}p^2a^2(\eta_{31}{}^2 + \eta_{32}{}^2)\}. \quad . \quad . \quad (23)$$

For a rectangular section $2a \times 2b$, the side $2a$ being parallel to the x-axis,

$$k = p(E_3/\rho)^{\frac{1}{2}}\{1 - \tfrac{1}{6}p^2(a^2\eta_{31}{}^2 + b^2\eta_{32}{}^2)\}. \quad . \quad . \quad (24)$$

For a given size and shape of rectangle, the correction to the first approximation is largest when the longer side is that answering to the larger Poisson's ratio (for traction along the rod). Possibly experimental use might be made of this result in examining materials for æolotropy.

If the material, though not isotropic, be symmetrical in structure round lines parallel to the length of the rod,

$$\eta_{31} = \eta_{32} = \eta, \text{ say,}$$

and writing E for E_3 in (22) we reproduce the result (60) of paper (C).

The results (22), (23), and (24), so far as my knowledge goes, are absolutely new.

Extension of Earlier Results.

§ 9. My paper (A), like the corresponding investigation of Pochhammer, dealt only with a solid circular cylinder; but the same method is applicable to a hollow circular cylinder. For greater continuity I shall employ in the remainder of this paper the notation of paper (A).

The displacements are u outwards along r, the perpendicular on the axis, and w parallel to the axis, taken as that of z. Thomson and Tait's notation m, n for the elastic constants is employed. The frequency is $k/2\pi$ and the density ρ, as in the earlier part of this paper, and for brevity

$$k^2\rho/(m+n) = \alpha^2, \quad k^2\rho/n = \beta^2, \quad . \quad . \quad . \quad (25)$$

so that α and β have utterly different significations from their previous ones.

There being no displacement perpendicular to r, in a transverse section, the dilatation δ is given by

$$\delta=\frac{du}{dr}+\frac{u}{r}+\frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

It was shown in paper (A) that the following equations held

$$\frac{d^2\delta}{dr^2}+\frac{1}{r}\frac{d\delta}{dr}+\frac{d^2\delta}{dz^2}+\alpha^2\delta=0, \quad . \quad . \quad . \quad . \qquad (27)$$

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+\frac{d^2u}{dz^2}+\beta^2u=-\frac{m}{n}\frac{d\delta}{dr}, \qquad (28)$$

$$\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}+\frac{d^2w}{dz^2}+\beta^2w=-\frac{m}{n}\frac{d\delta}{dz}. \quad . \qquad (29)$$

Employing J and Y to represent the two solutions of Bessel's equation we find, as in paper (A), that the above equations are satisfied by

$$\delta=\cos kt\cos(pz-\epsilon)\left\{CJ_0(r(\alpha^2-p^2)^{\frac{1}{2}})+C'Y_0(r(\alpha^2-p^2)^{\frac{1}{2}})\right\}, (30)$$

$$u=\cos kt\cos(pz-\epsilon)\Big[AJ_1(r(\beta^2-p^2)^{\frac{1}{2}})+A'Y_1(r(\beta^2-p^2)^{\frac{1}{2}})$$
$$-\frac{m}{n}\frac{(\alpha^2-p^2)^{\frac{1}{2}}}{\alpha^2-\beta^2}\left\{CJ_1(r(\alpha^2-p^2)^{\frac{1}{2}})+C'Y_1(r(\alpha^2-p^2)^{\frac{1}{2}})\right\}\Big], \quad . \qquad (31)$$

$$w=-\cos kt\sin(pz-\epsilon)\Big[\left\{AJ_0(r(\beta^2-p^2)^{\frac{1}{2}})+A'Y_0(r(\beta^2-p^2)^{\frac{1}{2}})\right\}$$
$$\times\frac{(\beta^2-p^2)^{\frac{1}{2}}}{p}+\frac{mp}{n(\alpha^2-\beta^2)}\left\{CJ_0(r(\alpha^2-p^2)^{\frac{1}{2}})+C'Y_0(r(\alpha^2-p^2)^{\frac{1}{2}})\right\}\Big], (32)$$

where A, A′, C, C′ are arbitrary constants to be determined by the surface conditions.

In reality α^2-p^2 is negative; but the properties of the J and Y Bessel functions which at present concern us are not affected thereby. β^2-p^2, on the other hand, is positive.

§ 10. If a and b are the radii of the outer and inner cylindrical surfaces respectively, then from the conditions which

hold over these surfaces we must have

$$\frac{du}{dz}+\frac{dw}{dr}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$(m-n)\delta+2n\frac{du}{dr}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

when $r=a$, and when $r=b$.

As regards the terminal conditions we should have, following the ordinary view of longitudinal vibrations,

$$w=0 \text{ over a fixed end}, \quad . \quad . \quad . \quad . \quad (35)$$

$$\left.\begin{aligned} \widehat{zz} &\equiv (m-n)\delta+2n\frac{dw}{dz}=0, \\ \widehat{zr} &\equiv n\left(\frac{du}{dz}+\frac{dw}{dr}\right)=0 \end{aligned}\right\} \text{over a free end.} \quad (36)$$

We have no means of satisfying these terminal equations by means of the present solution save by selecting suitable values for p and ϵ. Clearly if both ends $z=0$ and $z=l$ be fixed we accomplish our object by putting

$$\epsilon=0, \quad p=i\pi/l.$$

If, however, $z=l$ be a free end, while $z=0$ is a fixed, we must have $\epsilon=0$ to satisfy the conditions of the fixed end; and this leaves us with

$$\widehat{zz} \propto \cos pl,$$

$$\widehat{zr} \propto \sin pl$$

over the free end. This is the difficulty we have already indicated in § 4; and it is in no respect peculiar to hollow cylinders, and need not further concern us at present.

§ 11. In dealing with the surface conditions, brevity is effected by the use of the notation

$$\lambda=\sqrt{\alpha^2-p^2}, \qquad \mu=\sqrt{\beta^2-p^2}, \quad . \quad . \quad (37)$$

whence

$$\alpha^2-\beta^2=\lambda^2-\mu^2.$$

After simplifications, into which I need not enter, the elimination of A, A′, C, C′ from the four equations holding over the cylindrical surface supplies the determinantal equation

$$\begin{vmatrix} J_1(a\mu) & Y_1(a\mu) & -\dfrac{2p^2\lambda}{a^2(\mu^2-p^2)}J_1(a\lambda) & -\dfrac{2p^2\lambda}{a^2(\mu^2-p^2)}Y_1(a\lambda) \\ J_1(b\mu) & Y_1(b\mu) & -\dfrac{2p^2\lambda}{a^2(\mu^2-p^2)}J_1(b\lambda) & -\dfrac{2p^2\lambda}{a^2(\mu^2-p^2)}Y_1(b\lambda) \\ J_1'(a\mu) & Y_1'(a\mu) & \dfrac{m-n}{2n\mu}J_0(a\lambda)+\dfrac{\lambda^2}{a^2\mu}J_1'(a\lambda) & \dfrac{m-n}{2n\mu}Y_0(a\lambda)+\dfrac{\lambda^2}{a^2\mu}Y_1'(a\lambda) \\ J_1'(b\mu) & Y_1'(b\mu) & \dfrac{m-n}{2n\mu}J_0(b\lambda)+\dfrac{\lambda^2}{a^2\mu}J_1'(b\lambda) & \dfrac{m-n}{2n\mu}Y_0(b\lambda)+\dfrac{\lambda^2}{a^2\mu}Y_1'(b\lambda) \end{vmatrix}=0. \quad (38)$$

This equation is true irrespective of the *relative* magnitudes of a and b. It constitutes a frequency equation supplying values of k which apply to the type of vibrations consistent with the surface conditions. If both ends of the rod be fixed there is no restriction to the absolute values of a/l and b/l; but if one or both ends are free, such a restriction is really involved in the fact that unless ia/l and ib/l be both small—i being the order of the harmonic of the fundamental note under consideration—the failure to satisfy exactly the terminal condition

$$zr=0$$

involves an inconsistency which cannot be allowed.

§ 12. The case of a thick rod fixed at both ends is of little physical interest, and the treatment of (38) in its utmost generality would involve grave mathematical difficulties. I thus limit my attention to the case when ia/l and ib/l are both small. This implies that $a\lambda$, $b\lambda$, $a\mu$, and $b\mu$ are all small. Thus in dealing with the various Bessel functions we may use the following approximations *, which hold so long as the variable x is small,

$$\left.\begin{aligned} J_0(x)&=1-x^2/4, \\ J_1(x)&=\tfrac{1}{2}x(1-x^2/8), \\ J_1'(x)&=\tfrac{1}{2}(1-\tfrac{3}{8}x^2), \\ Y_0(x)&=(1-x^2/4)\log x+x^2/4, \\ Y_1(x)&=\frac{x}{2}(1-x^2/8)\log x-x^{-1}-x/4, \\ Y_1'(x)&=\tfrac{1}{2}(1-\tfrac{3}{8}x^2)\log x+x^{-2}+\tfrac{1}{4}. \end{aligned}\right\} \quad . \quad (39)$$

Retaining only the principal terms in (38), we are of course led at once to the first approximation (1). Again, if (1) held

* *Cf.* Gray and Mathews' 'Treatise on Bessel Functions,' pp. 11, 22, &c.

exactly we should find

$$\left.\begin{aligned} \alpha^2&=p^2n(3m-n)\div m(m+n), \\ \lambda^2&=-p^2(m-n)^2\div m(m+n), \\ \mu^2-p^2&=p^2(m-n)/m; \end{aligned}\right\} \quad . \quad . \quad (40)$$

and to a first approximation

$$-\frac{J_1(a\lambda)}{J_1(a\mu)}\frac{2p^2\lambda}{\alpha^2(\mu^2-p^2)}=-\frac{2p^2\lambda^2}{\alpha^2\mu(\mu^2-p^2)}=\frac{2}{\mu}\frac{m(m-n)}{n(3m-n)},$$

$$\frac{m-n}{2n\mu}\frac{J_0(a\lambda)}{J_1'(a\mu)}+\frac{\lambda^2}{\alpha^2\mu}\frac{J_1'(a\lambda)}{J_1'(a\mu)}=\frac{2(m-n)}{2n\mu}+\frac{\lambda^2}{\alpha^2\mu}=\frac{2m(m-n)}{\mu n(3m-n)},$$

and similarly if a be replaced by b.

We thus see that the third column in the determinant (38) is such that each principal term in it is obtained by multiplying the principal term in the same row in the first column by the same constant $2m(m-n)\div\{\mu n(3m-n)\}$. Now if one column of a determinant is obtainable by multiplying another column by a constant that determinant vanishes. It is thus at once clear that in proceeding even to a second approximation we need retain only the principal terms in the second and fourth columns of (38). This removes what seemed at first sight a formidable obstacle, viz. the occurrence of the logs in the expressions for Y_0, Y_1, and Y_1'.

§ 13. For further simplification of the determinant multiply each term by 2, and divide the first and second rows by $a\mu$ and $b\mu$ respectively. Then for the second row write the difference between the first and second rows, and for the fourth row the difference between the third and fourth rows, and multiply the resulting rows by $b^2/(a^2-b^2)$. Finally multiply the second column by $a^2\mu^2/2$, the third column by $\mu\alpha^2$, and the fourth column by $a^2\mu\alpha^2/2$. We thus reduce (38) to the easily manageable form

$$\begin{vmatrix} 1-\frac{1}{8}a^2\mu^2 & -1 & -\frac{2\lambda^2p^2}{\mu^2-p^2}(1-\frac{1}{8}a^2\lambda^2) & \frac{2p^2}{\mu^2-p^2} \\ \frac{1}{8}b^2\mu^2 & -1 & -\frac{\lambda^4p^2b^2}{4(\mu^2-p^2)} & \frac{2p^2}{\mu^2-p^2} \\ 1-\frac{3}{8}a^2\mu^2 & 1 & \lambda^2+\alpha^2\frac{m-n}{n}-\frac{a^2\lambda^2}{8}\left(3\lambda^2+2\alpha^2\frac{m-n}{n}\right) & 1 \\ \frac{3}{8}b^2\mu^2 & 1 & \frac{1}{8}b^2\lambda^2\left(3\lambda^2+2\alpha^2\frac{m-n}{n}\right) & 1 \end{vmatrix}=0. \quad (41)$$

After algebraic reduction, use being made in the secondary terms of the first approximation results (1), (40), &c., we easily deduce from (41)

$$k^2 = \frac{p^2}{\rho}\frac{n(3m-n)}{m}\left\{1-\tfrac{1}{2}p^2\left(\frac{m-n}{2m}\right)^2(a^2+b^2)\right\}, \quad . \quad (42)$$

whence

$$k = p(\mathrm{E}/\rho)^{\frac{1}{2}}\left\{1-\tfrac{1}{2}p^2\eta^2\frac{a^2+b^2}{2}\right\}. \quad . \quad . \quad . \quad (43)$$

In a hollow circular section the radius of gyration round the perpendicular to the plane through the centre is given by

$$\kappa^2 = (a^2+b^2)/2,$$

so that (43) is in agreement with (3) and (22).

The fact that (43) is merely a special case of (3) or (22) may seem to indicate that our separate treatment of the hollow cylinder, or tube, is quite unnecessary. I can only say that having regard to the methods by which (3) and (22) were arrived at—more especially to the fact that in establishing (6) I was dealing with solid cylinders—I had long felt the desirability of an independent investigation.

§ 14. The complete determination of the constants A, A′, C, C′, and of the several displacements, strains, and stresses to the degree of accuracy assumed in (43), though not a very arduous labour, would require more time than seems warranted by the physical interest of the problem. I thus confine my further remarks to the form of the longitudinal displacement w. Substituting their approximate values for the J's and Y's from (39) in (32), we find

$$\begin{aligned} pw/\mu \cos kt \sin(pz-\epsilon) = \\ -\mathrm{A}(1-\tfrac{1}{4}\mu^2r^2) - \mathrm{A}'\{(1-\tfrac{1}{4}\mu^2r^2)\log\mu r + \tfrac{1}{4}\mu^2r^2\} \\ + \frac{\mathrm{C}p^2}{\mu\alpha^2}(1-\tfrac{1}{4}\lambda^2r^2) + \frac{\mathrm{C}'p^2}{\mu\alpha^2}\{(1-\tfrac{1}{4}\lambda^2r^2)\log\lambda r + \tfrac{1}{4}\lambda^2r^2\}. \quad (44)\end{aligned}$$

Now, considering only their principal terms, it is easily seen that A′/A and C′/C are both of the order $(p^2ab)^2$. Thus, to the present degree of approximation, we may leave the A′ and C′ terms in w out of account. Also confining ourselves to principal terms, we easily find

$$\frac{-\mathrm{A}}{2(m-n)} = \frac{\mathrm{C}p^2/\mu\alpha^2}{m+n}.$$

Hence, employing the two last of equations (40), we deduce

$$\tfrac{1}{4}\{A\mu^2-(Cp^2/\mu\alpha^2)\lambda^2\}\div\{-A+Cp^2/\mu\alpha^2\}$$
$$=-\frac{m-n}{4m}p^2=-\tfrac{1}{2}\eta p^2.$$

We thus have from (44), to the degree of approximation reached in (43),

$$w=w_0\cos kt\sin(pz-\epsilon)(1-\tfrac{1}{2}\eta p^2r^2), \quad . \quad . \quad . \quad (45)$$

where w_0 is a constant which depends on the amplitude of the vibration.

The expression (45) for w is exactly the same as I found in my earlier papers for solid cylinders; the r^2 term representing one of the additions I deem it necessary to make to Lord Rayleigh's assumed type of vibration.

The paraboloidal form of cross section, met with except at nodes or when $\cos kt$ vanishes, seems to me an interesting feature of the longitudinal type of vibrations. Possibly, observations on light reflected from a polished terminal face might lead a skilled experimentalist to interesting conclusions as to the value of η.

It should, however, be borne in mind that, inasmuch as the terminal condition $\widehat{zr}=0$ is not exactly satisfied by the above solution in fixed-free vibrations, there may be a slight departure from the theoretical form in the immediate neighbourhood of a free end.

§ 15. The result (43) is true irrespective of the relative magnitudes of b and a. If b/a be very small, the correctional term is the same as for a solid cylinder of the same external radius. If, on the other hand, b/a be very nearly unity, or the cylinder take the form of a thin-walled tube, we have

$$k=p(E/\rho)^{\frac{1}{2}}(1-\tfrac{1}{2}p^2\eta^2a^2). \quad . \quad . \quad . \quad . \quad (46)$$

The correctional term is here twice as great as in a solid cylinder of radius a.

§ 16. An experimental investigation into the influence of the shape and dimensions of the cross-section on the frequency of longitudinal vibrations is certainly desirable. In comparing the results of such an investigation with the theoretical results here determined, several considerations must, however, be borne in mind.

Statical and dynamical elastic moduli are to some exten different, so that the value of E occurring in (2) or (3) is not that directly measured by statical experiments *. In other words the difference between the observed frequency of the fundamental vibration in a fixed-free bar, and the frequency calculated from the ordinary formula

$$k/2\pi = (1/4l)\sqrt{E/\rho},$$

if E be determined directly by statical experiments, is not to be wholly attributed to the defect of the ordinary first approximation formula. Again, it must be remembered that E varies †, often to a very considerable extent, in material nominally the same; so that the difference of pitch observed in different rods cannot without further investigation be safely ascribed to differences in the area or shape of their cross-sections. Further, elastic moduli may alter under mechanical treatment, so that it would be unsafe to assume if a hollow bar were further hollowed or were altered in shape, that its Young's modulus would remain unaffected.

If it were possible to measure with sufficient accuracy the frequency of the fundamental note and several of its "harmonics" in a single rod, one would have a more certain basis of comparison with the theoretical results. Even in this case, however, there is the consideration that in practice the rod must be supported in some way, and this is likely to introduce some constraint not accurately represented by the theoretical conditions. Again, reaction between the vibrating rod and the surrounding medium may not be absolutely without influence on the pitch ‡.

I mention these difficulties because their recognition may prevent a considerable waste of time on the part of anyone engaged in experiments on the subject.

Though somewhat of a side issue, it may be worth remarking that the correction factor $1-\frac{1}{2}p^2\eta^2\kappa^2$ for the frequency in

* See Lord Kelvin's Encyclopædia article on Elasticity, § 75, or Todhunter and Pearson's 'History,' vol. iii. art. 1751.

† For the effects of possible variation in the material throughout the bar, see the Phil. Mag. for Feb. 1886, pp. 81–100.

‡ *Cf.* Lamb, Memoirs and Proceedings Manchester Phil. Society, vol. xlii. part iii. 1898.

isotropic material contains no elastic constant except Poisson's ratio. Thus observations made on rods differing only in material might throw some light on the historic question whether η is or is not the same for all isotropic substances.

The discussion of equations (6) and of the experimental side of the problem has been largely expanded at the suggestion of the Society's referee.

XXXIII. *On the Thermal Properties of Normal Pentane.* *By* J. ROSE-INNES, *M.A.*, *B.Sc.*, *and* Prof. SYDNEY YOUNG, *D.Sc.*, *F.R.S.**

IN the year 1894 an experimental investigation of the relations between the temperatures, pressures, and volumes of Isopentane, through a very wide range of volume, was carried out by one of us, and the results were published in the Proc. Phys. Soc. xiii. pp. 602–657. It was there shown that the relation $p=bT-a$ at constant volume (where a and b are constants depending on the nature of the substance and on the volume) holds good with at any rate but small error from the largest volume (4000 cub. cms. per gram) to the smallest (1·58 cub. cms. per gram).

In the neighbourhood of the critical volume (4·266 cub. cms.), and at large and very small volumes, the observed deviations were well within the limits of experimental error, but at intermediate volumes they were somewhat greater, and, as they exhibited considerable regularity, it is a question whether they could be attributed entirely to errors of experiment. In any case, the relation may be accepted as a close approximation to the truth.

A quantity of pure normal pentane having been obtained by the fractional distillation of the light distillate from American petroleum, it was decided to carry out a similar investigation with this substance; but, as it had been found

* Read December 9, 1898.

that isopentane vapour at the largest volumes behaves practically as a normal gas, it was not considered necessary to make the determinations through so wide a range of volume.

The method employed for the separation of the normal pentane from petroleum has been fully described in the Trans. Chem. Soc. 1897, lxxi. p. 442; and the vapour-pressures, specific volumes as liquid and saturated vapour, and critical constants have been given in the same journal (p. 446).

The data for the isothermals of normal pentane were obtained by precisely the same experimental methods as in the case of isopentane, and reference need, therefore, only be made to the previous paper (*loc. cit.*).

There were four series of determinations; and particulars as to the mass of pentane, and the data obtained in each series are given below:—

Series.	Mass of Normal Pentane.	Data obtained.
	gram.	
I.	·10922	Volumes of liquid to critical point; volumes above critical temperature to 280°.
II.	·02294	Volumes of unsaturated vapour from 140° to critical point; volumes above critical point to 280°.
III.	·005858	Volumes of vapour at and above 80°.
IV.	·001845	Volumes of vapour at and above 40°.

The correction for the vapour-pressure of mercury was made in the same way as with isopentane: when liquid was present it was assumed that the mercury vapour exerted no pressure; in Series I., above the critical point, one-fourth of the maximum vapour-pressure of mercury was subtracted; in Series II. one-half; in Series III. three-fourths; and in Series IV. the full pressure.

The volumes of a gram of liquid and unsaturated vapour are given in the following tables.

Volumes of a Gram of Liquid and of Unsaturated Vapour.

SERIES 1.

Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.
130°.	7023*	2·022	170°. (*cont.*)	29270	2·269	200°. (*cont.*)	26910	3·329
	8194	2·018		31990	2·250		27670	3·136
	11430	2·008		35040	2·230		29350	2·944
	15100	1·998		38270	2·211		30820	2·847
	18610	1·989		42170	2·191		33060	2·751
	22630	1·979		46180	2·172		36260	2·655
	26810	1·969		50760	2·154		41090	2·558
	31010	1·959		55660	2·135		46560	2·482
	35560	1·950	180°.	19340	2·580		49980	2·443
	40280	1·940		20110	2·557		51920	2·424
	45520	1·930		20770	2·538		54060	2·405
	50450	1·921		21570	2·519	210°.	28410	6·055
140°.	8507*	2·094		22470	2·500		28790	5·666
	9927	2·089		23420	2·481		29180	5·274
	10580	2·085		24570	2·462		29530	4·883
	13230	2·075		25730	2·443		29920	4·492
	15730	2·066		27200	2·423		30400	4·106
	18280	2·056		28550	2·404		31160	3·716
	21300	2·046		30230	2·384		32660	3·330
	24140	2·036		32060	2·365		34200	3·137
	27300	2·027		33970	2·346		36870	2·945
	30670	2·018		36240	2·327		39010	2·848
	34610	2·008		38630	2·307		41970	2·752
	38340	1·999		41100	2·288		46070	2·656
	42710	1·989		44280	2·269		49360	2·598
	46880	1·979		47430	2·250		51950	2·559
150°.	11380*	2·172		50830	2·231		54940	2·520
	14680	2·153		54400	2·211	220°.	31010	6·056
	18770	2·133	190°.	22490	2·900		31610	5·667
	22830	2·114		22750	2·866		32270	5·275
	27840	2·095		23180	2·827		32920	4·884
	33180	2·075		23690	2·789		33670	4·493
	39100	2·056		24310	2·750		34630	4·107
	45700	2·037		25110	2·712		36030	3·717
	53080	2·018		26100	2·673		38540	3·331
160°.	14060	2·272		27230	2·635		40790	3·138
	14330	2·268		28630	2·596		44510	2·946
	16670	2·249		30340	2·558		47320	2·849
	19250	2·229		32340	2·520		51110	2·752
	22000	2·210		34780	2·481		56010	2·657
	25350	2·191		37650	2·443	230°.	33630	6·058
	29010	2·172		41060	2·404		34420	5·669
	32900	2·153		43110	2·385		35320	5·277
	37460	2·134		45310	2·366		36320	4·885
	42200	2·115		47620	2·346		37460	4·495
	47860	2·095		50200	2·327		38930	4·108
	53780	2·076		52950	2·308		40960	3·718
170°.	16560	2·398	200°.	25720	6·053		42530	3·525
	17520	2·384		25890	5·665		44530	3·331
	19020	2·364		26030	5·273		47610	3·139
	20660	2·345		26100	4·881		49680	3·042
	22520	2·326		26190	4·491		52410	2·946
	24460	2·307		26250	4·105		55850	2·850
	26630	2·288		26400	3·715	240°.	36240	6·059

* Pressure below vapour-pressure.

Series I. (*continued*).

Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.
240°	37310	5·670	250°.	45160	4·497	270°.	43780	6·064
(*cont.*)	38430	5·278	(*cont.*)	47720	4·110		45570	5·675
	39770	4·886		49390	3·915		47600	5·282
	41350	4·496		51370	3·720		49990	4·890
	43310	4·109		53890	3·527		51320	4·695
	46170	3·719	260°.	41260	6·062		52890	4·499
	48150	3·526		42850	5·673		54600	4·306
	50870	3·332		44590	5·281	280°.	46310	6·065
	54160	3·140		46530	4·889		48350	5·676
250°.	38810	6·061		49020	4·498		50620	5·283
	40110	5·672		50420	4·304		51960	5·087
	41510	5·279		52190	4·111		53300	4·891
	43230	4·888		54090	3·916		54930	4·696
							56810	4·500

Series II.

Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.
140°.	8866	30·61	200°.	18460	14·93	240°.	14570	25·13
	9055	29·70	(*cont.*)	19160	14·01	(*cont.*)	15530	23·27
	9255	28·77		19880	13·10		16600	21·41
	9466	27·85		20660	12·18		17810	19·56
	9682	26·93		21470	11·27		19220	17·71
	9886	26·00		22300	10·35		20380	16·32
160°.	9822	29·71		23130	9·44		21730	14·95
	10570	26·94		23950	8·52		22710	14·03
	11100	25·08		24720	7·61		23750	13·11
	11700	23·22		25330	6·69		24910	12·19
	12340	21·36		25620	6·23		26160	11·28
	13040	19·52	220°.	11960	29·76		27560	10·36
	13420	18·60		12930	26·98		29070	9·45
	13790	17·67		13720	25·11		30700	8·53
180°.	10550	29·73		14580	23·25		32570	7·61
	11380	26·96		15560	21·39		34670	6·70
	12000	25·09		16660	19·55		35790	6·24
	12690	23·23		17900	17·70		36990	5·78
	13450	21·37		18950	16·31		38330	5·32
	14280	19·53		20100	14·94		39830	4·87
	15200	17·68		20950	14·02		41730	4·41
	15700	16·76		21850	13·10	260°.	13340	29·79
	16220	15·84		22810	12·19		14500	27·01
	16770	14·93		23850	11·27		15410	25·14
	17320	14·01		24950	10·35		16440	23·28
	17890	13·09		26140	9·44		17610	21·42
	18460	12·17		27380	8·52		18940	19 57
	19010	11·26		28720	7·61		20480	17·72
200°.	11260	29·75		30080	6·69		21810	16·33
	12160	26·97		30720	6·23		23300	14·95
	12870	25·10		31440	5·78		24410	14·04
	13640	23·24		32160	5·32		25620	13·12
	14510	21·38		32990	4·86		26960	12·20
	15470	19·54		33900	4·40		28460	11·28
	16570	17·69	240°.	12650	29·77		30090	10·36
	17480	16·30		13740	27·00		31900	9·45

Series II. (*continued*).

Temp.	Pressure millim.	Volume. cub. cm.	Temp.	Pressure millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.
260°.	33970	8·53	280°.	16210	25·15	280°.	30600	11·29
(*cont.*)	36340	7·62	(*cont.*)	17320	23·29	(*cont.*)	32550	10·37
	39120	6·70		18580	21·43		34680	9·46
	40630	6·24		20030	19·58		37130	8·54
	42360	5 78		21710	17·73		39980	7·62
	44370	5·32		23160	16·34		43400	6·70
	46640	4·87		24810	14·96		45350	6·24
	49650	4·41		26050	14·05		47680	5·79
280°.	13980	29·80		27410	13·12		50280	5·32
	15230	27·02		28940	12·21		52150	5·05
							54120	4·78

Series III.

Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.
80°.	2371	116·11	140°	8325	33·32	200°	13140	24·40
	2441	112·50	(*cont.*)	9040	29·74	(*cont.*)	13930	22·62
	2510	108·90		9435	27·95	240°.	3631	116·59
	2586	105·29		9860	26·16		3988	105·72
	2666	101·64	160°.	3015	116·35		4276	98·40
	2708	99·83		3301	105·50		4603	91·10
100°.	2537	116·17		3532	98·20		4980	83·83
	2770	105·34		3795	90·91		5428	76·61
	2955	98·05		4092	83·65		5958	69·34
	3166	90·77		4448	76·45		6426	63·91
	3405	83·52		4859	69·20		6970	58·54
	3681	76·33		5231	63·78		7372	54·95
	4006	69·09		5652	58·42		7823	51·34
	4190	65·50		5979	54·83		8362	47·75
	4398	61·90		6324	51·23		8965	44·16
120°.	2701	116·23		6733	47·65		9677	40·57
	2951	105·39		7192	44 07		10500	37·00
	3152	98·10		7685	40·49		11480	33·41
	3376	90·82		8284	36·92		12650	29·82
	3638	83·56		8977	33·34		13330	28·02
	3940	76·37		9795	29·76		14090	26·23
	4294	69·13		10260	27·97		14920	24·42
	4607	63·71		10770	26·17		15870	22·64
	4960	58·36		11310	24·37	280°.	3933	116·71
	5234	54·78		11890	22·59		4337	105·83
	5531	51·18	200°.	3317	116·47		4656	98·50
	5859	47·60		3642	105·61		5022	91·19
	6233	44·02		3896	98·30		5440	83·92
	6639	40·44		4190	91·01		5936	76·68
140°.	2861	116·29		4533	83·74		6511	69·41
	3129	105·45		4927	76·53		7024	63·98
	3342	98·15		5401	69·27		7634	58·60
	3587	90 87		5812	63·85		8074	55·00
	3870	83 60		6299	58·48		8596	51·39
	4197	76·41		6665	54·89		9187	47·80
	4577	69·16		7084	51·29		9878	44·20
	4920	63·75		7554	47·70		10670	40·61
	5316	58·39		8066	44·12		11600	37 04
	5607	54·80		8678	40·53		12700	33·44
	5943	51·21		9387	36·96		14050	29·85
	6309	47·63		10240	33·37		14830	28·05
	6717	44·05		11240	29·79		15690	26·25
	7179	40·47		11810	27 99		16680	24·45
	7708	36·90		12440	26·20		17800	22·66

Series IV.

Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.	Temp.	Pressure. millim.	Volume. cub. cm.
40°.	858	299·3	120°.	928	357·6	200°	2415	162·8
	869	293·5		989	334·6	(*cont.*)	2588	151·5
60°.	857	322·6		1061	311·5		2788	140·1
	889	311·0		1143	288·4		3024	128·7
	956	287·9		1237	265·3		3302	117·4
	1034	264·9		1350	242·5		3458	111·6
	1125	242·1		1485	219·5		3632	106·0
	1236	219·1		1604	202·3		3830	100·2
	1299	207·7		1745	185·3		4049	94·6
	1370	196·3		1854	173·9	240°.	1217	358·7
	1447	185·0		1977	162·5		1299	335·7
	1534	173·7		2116	151·1		1394	312·4
	1580	168·0		2276	139·8		1502	289·3
80°.	882	334·3		2463	128·4		1631	266·1
	946	311·2		2680	117·1		1782	243·2
	1019	288·1		2807	111·4		1964	220·2
	1102	265·0		2946	105·7		2126	202·9
	1200	242·2		3096	100·0		2318	185·9
	1319	219·3		3264	94·4		2464	174·5
	1424	202·1	160°.	1025	357·9		2629	163·0
	1548	185·1		1093	335·0		2821	151·6
	1642	173·7		1175	311·8		3042	140·2
	1748	162·3		1264	288·7		3301	128·8
	1868	151·0		1371	265·6		3609	117·5
	2006	139·6		1497	242·7		3783	111·7
	2167	128·3		1647	219·7		3975	106·1
	2356	117·0		1783	202·5		4191	100·3
	2463	111·3		1939	185·5		4430	94·7
	2579	105·6		2061	174·1	280°.	1314	359·0
	2705	100·0		2198	162·7		1404	336·0
100°.	892	351·7		2357	151·3		1507	312·8
	936	334·5		2535	139·9		1626	289·5
	1002	311·3		2747	128·5		1765	266·4
	1081	288·2		2996	117·2		1928	243·5
	1171	265·2		3140	111·5		2124	220·4
	1275	242·3		3295	105·9		2301	203·1
	1402	219·4		3468	100·1		2509	186·0
	1516	202·2		3664	94·5		2670	174·6
	1647	185·2	200°.	1121	358·3		2850	163·2
	1749	173·8		1195	335·3		3056	151·8
	1862	162·4		1283	312·1		3296	140·4
	1993	151·1		1382	289·0		3578	128·9
	2141	139·7		1501	265·9		3917	117·6
	2316	128·4		1639	243·0		4104	111·9
	2519	117·0		1805	219·9		4321	106·2
	2638	111·3		1953	202·7		4555	100·4
	2763	105·7		2128	185·7		4815	94·8
	2902	100·0		2264	174·3			
	3058	94·3						

Relation of Pressure to Temperature at Constant Volume. Isochors.

For the smaller volumes isobars were first constructed from the isothermals, and the temperatures at definite volumes were read from the isobars. The data from which the isobars were constructed are given below :—

Isobars read from Isothermals.

Temp....	130°.	140°.	150°.	160°.	170°.	180°.	190°.	200°.	210°.	220°.	230°.
Pressure in metres.	Volume in cub. cms.										
12	2·0066	2·0796	2·1692								
16	1·9958	2·0639	2·1464	2·2546							
20	1·9852	2·0500	2·1266	2·2240	2·3526	2·5600					
24	1·9755	2·0372	2·1094	2·1986	2·3115	2·4715	2·7700				
28	1·9661	2·0254	2·0935	2·1767	2·2780	2·4105	2·6145	3·0840			
32	1·9574	2·0144	2·0790	2·1574	2·2496	2·3650	2·5255	2·7915			
36	1·9486	2·0042	2·0656	2·1398	2·2247	2·3285	2·4635	2·6602	2·9910		
40	1·9406	1·9948	2·0532	2·1256	2·2026	2·2965	2·4160	2·5775	2·8100	3·1960	
44	1·9330	1·9861	2·0417	2·1087	2·1830	2·2695	2·3772	2·5135	2·7000	2·9660	
48	1·9257	1·9776	2·0307	2·0946	2·1652	2·2460	2·3438	2·4630	2·6180	2·8290	3·1180
52	1·9186	...	2·0204	2·0816	2·1487	2·2245	2·3150	2·4225	2·5575	2·7330	2·9590
56	...	...	2·0106	2·0691	2·1335	...	...	...	...	2·6570	2·8460

In the following tables the data for the isochors are given ; those for small volumes were read from the isobars, and those for larger volumes from the isotherms.

Isochors read from Isobars.

Volume.	2·0.	2·1.	2·2.	2·3.	2·4.	2·5.	2·6.	2·7.	2·8.	2·9.	3·0.
Pressure in metres.	Temperature.										
12	129·0	142·5									
16	130·7	144·7	155·35	163·45	169·6						
20	132·4	146·8	157·8	166 3	172·85	177·75					
24	134·15	148·8	160·1	169·1	176·0	181·3	185·35	188·4	190·6	192·1	193·1
28	136·0	150·8	162·5	171·85	179·25	185·05	189·45	192·55	195·0	197·25	198·9
32	137·7	152·9	165·0	174·65	182·4	188·65	193·5	197·4	200·15	...	...
36	139·4	154·9	167·2	177·5	185·75	192·05	197·4	201·5	204·95	207·8	210·2
40	141·0	156·8	169·8	180·4	188·85	195·7	201·25	205·85	209·7	212·75	215·6
44	142·9	158·8	172·05	183·05	191·8	199·05	205·1	210·0	214·3	217·9	220·9
48	144·55	160·95	174·65	185·9	195·0	202·6	208·9	214·25	218·8	222·75	226·3
52	146·3	162·95	176·8	188·3	198·0	206·0	212·75	218·3	223·2	227·7	231·5
56	...	164·95	...	...	...	...	...	222·4	227·75	232·4	...

Isochors read from Isothermals.

Volume.	2·9.	3·0.	3·2.	3·4.	3·6.	3·8.	4·0.	4·3.	4·6.	5·0.
Temp.	Pressure.									
190°.....	22500									
200......	29920	28720	27330	26760	26500	26360	26290	26200	26150	26090
210......	37770	35950	33600	32260	31480	30960	30580	30140	29800	29410
220......	45800	43210	39960	37930	36600	35660	34960	34120	33450	32720
230......	53940	50810	46430	43730	41820	40420	39400	38150	37140	36000
240......	......	......	53040	49870	47390	45450	44000	42250	40890	39370
250......	......	......	......	......	52830	50470	48630	46400	44630	42720
260......	......	......	......	......	......	......	53240	50550	48290	45940
270......	......	......	......	......	......	......	......	54670	52050	49260
280......	......	......	......	......	....	......	......	......	55800	52500

Isochors read from Isothermals.

Volume.	5·5.	6.	6·5.	7.	8.	9.	10.	12.	14.	16.
Temp.	Pressure.									
180°......	......	......	......	......	. ....	......	......	18600	17335	16130
190......	......	. ....	......	......	......	......	......	......	......	......
200......	25980	25760	25450	25140	24400	23540	22630	20840	19150	17690
210......	28940	28460	......	......	......	......	......	......	......	......
220......	31920	31090	30350	29630	28120	26730	25400	23000	20985	19200
230......	34800	33750	......	......	......	......	......	......	......	......
240......	37790	36400	35140	33940	31750	29830	28110	25150	22730	20680
250......	40740	39000	......	......	......	......	......	......	......	......
260......	43590	41510	39770	38140	35300	32870	30790	27260	24460	22140
270......	46400	44040	......	......	......	......	......	......	......	......
280......	49330	46600	44270	42220	38740	35860	33390	29270	26110	23555

Isochors read from Isothermals.

Volume.	18.	20.	22.	26.	30.	35.	40.	50.	60.	70.	80.
Temp.	Pressure.										
100°.....	......	......	......	......	......	.....	......	......	......	3965	3532
120......	.. ...	......	......	......	......	... ..	6725	5640	4849	4249	3782
140......	......	......	......	9880	8985	8025	7250	6055	5190	4535	4026
160......	13640	12865	12115	10825	9735	8635	7772	6465	5515	4812	4263
180......	15030	14060	13180	11695	10475	......	......	......	....	......	......
200......	16370	15210	14215	12520	11180	9830	8775	7250	6150	5345	4726
220......	17695	16370	15220	13330	11880	......	......	......	......	......	......
240......	18980	17505	16250	14160	12570	11020	9800	8010	6805	5900	5208
260......	20225	18610	17230	14980	13260	......	......	......	......	......	......
280......	21465	19690	18200	15790	13975	12200	10825	8815	7462	6465	5699

Isochors read from Isothermals.

Volume.	90.	100.	120.	140.	160.	180.	200.	230.	260.	300.	350.
Temp.	Pressure.										
40°......	......	......	......	......	......	......	......	......	......	857	......
60......	......	......	......	......	......	1485	1347	1182	1053	920	......
80......	......	2705	2304	2001	1773	1589	1440	1260	1123	981	......
100......	3187	2907	2464	2137	1889	1693	1532	1340	1193	1040	896
120......	3404	3098	2620	2272	2005	1794	1622	1419	1262	1100	947
140......	3613	3285	......	......	......	......	......	......	......	......	......
160......	3832	3477	2931	2534	2233	1996	1804	1576	1400	1219	1048
200......	4237	3832	3231	2791	2458	2196	1981	1729	1535	1334	1146
240......	4653	4212	3536	3047	2679	2393	2157	1882	1669	1451	1246
280......	5087	4588	3836	3305	2905	2592	2337	2038	1808	1571	1349

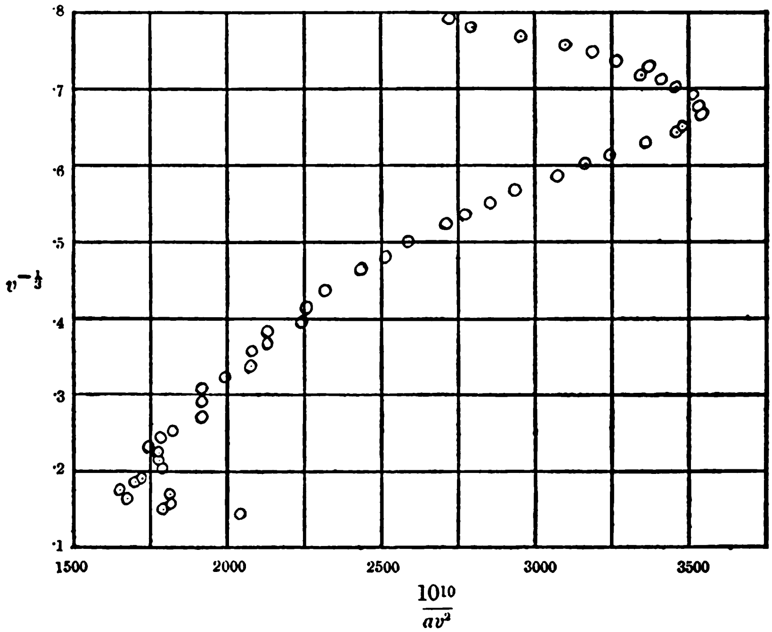
The values of b and a in the equation $p=b\mathrm{T}-a$ were obtained graphically from the preceding data. As with isopentane, the deviations are exceedingly small at the largest and smallest volumes and about the critical volume, but are larger at intermediate volumes; they exhibit a similar regularity and are in the same direction as with isopentane. Here again the relation $p=b\mathrm{T}-a$ at constant volume, if not absolutely true, may be taken as a very close approximation to the truth.

In studying the variation of b and a with the volume it was found convenient, in the case of isopentane, to plot the values of $\frac{10{,}000}{bv}$ and of $\frac{10^{10}}{av^2}$ against $v^{-\frac{1}{3}}$; and this has also been done for normal pentane. The values of b, a, $v^{-\frac{1}{3}}$, $\frac{10^4}{bv}$, and $\frac{10^{10}}{av^2}$ for a series of volumes are given in the table below and, for the sake of comparison, the corresponding values of $\frac{10^4}{bv}$ and $\frac{10^{10}}{av^2}$ for isopentane are added. (Table p. 331.)

The values of $10^{10}/av^2$ are plotted against $v^{-\frac{1}{3}}$ in the diagram on p. 332.

In a former paper by one of the authors (Phil. Mag. xliv. p. 77) it was pointed out that, besides the quantities b and a, it is often useful to consider a fresh quantity τ, which is defined as follows:—For each volume there is one and only one temperature at which the gas has its pressure equal to that given by the laws of a perfect gas; this temperature is denoted by τ. It is also shown that the numerical value

Vol. in c.c.'s.	b.	a.	$v^{-\frac{1}{3}}$.	$10^4/bv$.		$10^{10}/av^2$.	
	From drawn isochors.			N. Pentane.	Iso-pentane.	N. Pentane.	Iso-pentane.
2·0	2312	917,550	·7937	2·163	2·165	2725	2783
2·1	1980	811,210	·7809	2·405	2·487	2795	2952
2·2	1668	698,520	·7689	2·725	2·764	2958	3053
2·3	1436	610,860	·7576	3·028	3·051	3095	3181
2·4	1265	544,000	·7468	3·294	3·365	3191	3329
2·5	1132	490,380	·7368	3·534	3·602	3263	3397
2·6	1010	438,880	·7272	3·808	3·851	3371	3480
2·7	940	409,900	·7181	3·940	4·056	3347	3522
2·8	858	373,910	·7095	4·162	4·279	3411	3586
2·9	790	343,580	·7013	4·365	4·520	3461	3670
3·0	730	316,470	·6931	4·566	4·731	3511	3728
3·2	642·6	276,720	·6786	4·863	4·986	3529	3708
3·4	572·7	244,270	·6650	5·136	5·281	3541	3736
3·6	523·8	221,500	·6526	5·303	5·455	3484	3682
3·8	478·6	200,200	·6408	5·499	5·507	3459	3632
4·0	448·6	186,100	·6300	5·573	5·733	3358	3553
4·3	407·3	166,640	·6150	5·710	5·859	3245	3426
4·6	371·2	149,540	·6013	5·856	5·973	3160	3313
5·0	331·1	130,570	·5848	6·040	6·114	3069	3184
5·5	292·7	112,470	·5665	6·212	6·298	2939	3062
6·0	260·4	97,320	·5503	6·400	6·489	2855	2976
6·5	234·7	85,350	·5358	6·555	6·628	2773	2881
7	212·6	75,210	·5227	6·720	6·786	2713	2816
8	179·5	60,377	·5000	6·964	7·046	2588	2697
9	153·9	49,178	·4805	7·220	7·294	2510	2615
10	134·5	40,923	·4642	7·435	7·541	2444	2564
12	107·5	30.043	·4368	7·752	7·917	2312	2466
14	88·35	22,639	·4149	8·085	8·258	2254	2405
16	74·15	17,388	·3968	8·429	8·463	2247	2327
18	65·15	14,478	·3816	8·524	8·698	2132	2288
20	56·90	11,721	·3684	8·787	8·872	2133	2242
22	51·00	9,938	·3569	8·913	8·979	2079	2172
26	41·50	7,125	·3375	9·268	9·314	2076	2160
30	35·35	5,570	·3218	9·430	9·470	1995	2079
35	29·78	4,261	·3057	9·594	...	1916	...
40	25·47	3,263	·2924	9·815	9·813	1915	1969
50	19·73	2,089	·2714	10·14	10·26	1915	2067
60	16·23	1,521	·2554	10·27	10·46	1826	2046
70	13·75	1,143	·2426	10·39	10·53	1785	1957
80	11·91	894	·2321	10·50	10·49	1748	1788
90	10·45	694	·2231	10·63	10·61	1779	1782
100	9·325	563	·2154	10·72	10·74	1775	1845
120	7·653	388	·2027	10·89	10·87	1790	1823
140	6·520	296	·1926	10·96	10·92	1720	1747
160	5·677	230	·1842	11·01	11·07	1700	1860
180	5·030	187	·1771	11·04	11·18	1650	1930
200	4·478	138	·1710	11·17	11·24	1810	1950
230	3·892	113	·1632	11·15	11·18	1670	1730
260	3·418	81·5	·1567	11·25	11·22	1810	1700
300	2·953	62·0	·1496	11·29	11·27	1790	1680
350	2·513	40·0	·1419	11·37	11·33	2040	1770



of τ is given by the expression $\frac{a}{b - R/v}$; making use of the values of a and b already given for normal pentane, the values of τ have been calculated, and the results are given in the following Table :—

v.	τ.	v.	τ.	v.	τ.
2·0	488·0	4·6	814·9	35	833·9
2·1	517·1	5·0	824·3	40	841·0
2·2	547·6	5·5	828·8	50	848·0
2·3	576·0	6·0	835·4	60	826·6
2·4	601·0	6·5	838·4	70	810·6
2·5	623·4	7	843·2	80	798·2
2·6	647·4	8	843·3	90	813·6
2·7	660·9	9	848·9	100	817·1
2·8	680·3	10	850·1	120	845·3
2·9	698·0	12	844·9	140	843·3
3·0	715·8	14	849·2	160	821·4
3·2	742·3	16	861·6	180	806·0
3·4	766·5	18	843·2	200	862·5
3·6	780·2	20	854·3	230	824·8
3·8	796·7	22	845 8	260	840·2
4·0	799·7	26	859·5	300	826·7
4·3	807·0	30	849·1	350	869·6

An examination of this table shows that τ remains fairly constant for all large volumes down to about vol. 8. The actual numbers obtained vary a good deal; but these variations are sometimes in one direction and sometimes in another, and there is no steady increase or decrease. It appears, then, that all the values of τ above vol. 8 could be treated as the same without introducing any serious error; this occurred likewise in the case of isopentane. What is still more noteworthy is that the same constant value of τ could be used for both normal pentane and isopentane, keeping within the limits of experimental error. The mean value of τ for all volumes above 8 was found to be 842·4 for isopentane; it is 838·5 for normal pentane; and the intermediate value 840 could be used in both cases without introducing any error greater than the unavoidable errors of experiment.

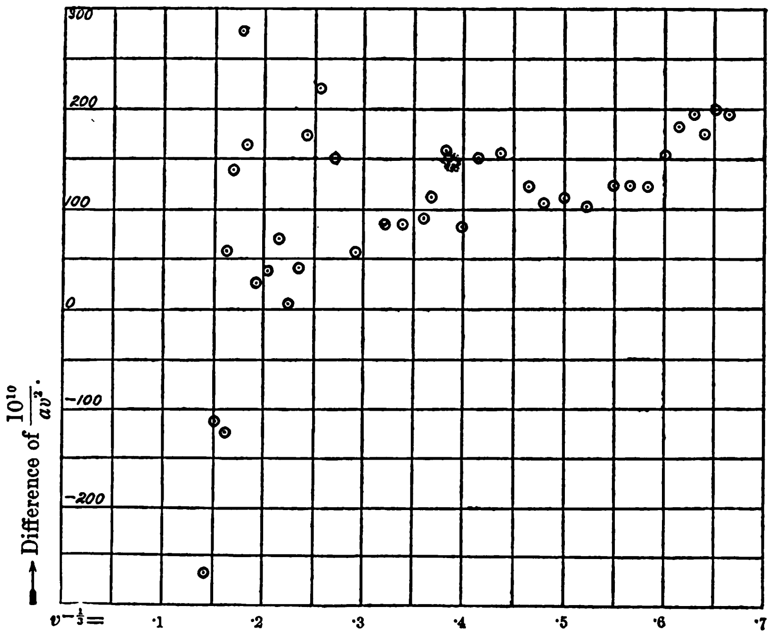
When we pass on to the neighbourhood of the critical point, the value of τ diminishes steadily as the volume decreases. For the critical volume itself τ is about 807, and for vol. 2 it has sunk to 488.

The most important conclusion arrived at in the case of isopentane was that the molecular pressure a does not follow a continuous law, but passes abruptly from one law to another somewhere about vol. 3·4 (Phil. Mag. xliv. p. 79). This inference was based on the study of a diagram in which the quantity $\frac{1}{av^2}$ was plotted against $v^{-\frac{1}{3}}$, and there appeared to be considerable evidence of discontinuity in the neighbourhood of the volume already mentioned. Of course it is impossible to *prove* discontinuity of slope by means of a series of isolated points, but it is suggested very strongly; and even if there be not discontinuity in the true mathematical sense of the term, there seems to be such a rapid change of behaviour as to amount practically to the same thing.

It was therefore a matter of some interest to discover whether the diagram obtained by plotting $\frac{1}{av^2}$ against $v^{-\frac{1}{3}}$ in the case of normal pentane would exhibit the same peculiarity. The diagram is given on p. 332, and it is easily seen that we have here a similar suggestion of discontinuity in the slope of $\frac{1}{av^2}$; this occurs somewhere about vol. 3·4, as with isopentane.

In attempting to find a formula for the pressure of normal pentane we are therefore confronted with the possibility that we may require two distinct algebraic equations. We may simplify the problem considerably by confining our attention to volumes lying above 3·4; and this limitation still leaves us with all those conditions of the substance in which we can most usefully compare it with isopentane.

Looking at the table on p. 331, which gives the series of values of $\frac{1}{av^2}$, and comparing it with the similar table for isopentane (Proc. Phys. Soc. xiii. pp. 654, 655), we notice that at the same volume the value of $\frac{1}{av^2}$ is always smaller in the former case than in the latter. The difference is not great, but it remains too persistently with the same sign for us to disregard it. As we proceed to larger and larger volumes,



however, the difference diminishes on the whole, and an interesting question arises whether we should be justified in treating it as ultimately vanishing when v is made infinite.

To elucidate this point a diagram was drawn in which the differences of $\frac{1}{av^2}$ between isopentane and normal pentane were plotted against $v^{-\frac{1}{3}}$; this diagram is reproduced, p. 334. The diminution in the differences with increase of volume is well shown in spite of the "wobbling" at large volumes; and a line running through the points might apparently end at zero difference. But though this result might be accepted as consistent with the experimental evidence if there were independent grounds for believing in it, it cannot be considered as the most probable judging solely from the diagram; we should be rather led to believe that even at infinite volumes the value of $\frac{1}{av^2}$ for isopentane remained larger than that for normal pentane.

The above results respecting a and τ are chiefly interesting because they seem capable of throwing some light on the vexed question of the influence exerted by difference of chemical structure on the thermal properties of a substance. Concerning this matter very little is known at present; but it is common knowledge among organic chemists that two substances may have the same chemical composition and show practically the same behaviour whilst in the condition of rare vapour, and yet they may differ considerably as to their thermal properties in the liquid state. The great field of observation in which the substances lie between the conditions of a rare vapour and a common liquid has been left almost entirely unexplored. This gap in our knowledge makes it impossible to say in what precise manner the difference between two isomeric substances originates; whether it arises conjointly with the first deviations from Boyle's law, or whether the difference remains inappreciable even with increasing density until we reach the neighbourhood of the critical point. We may put the problem more precisely as follows:—If we imagine the pressure given by a series of ascending powers of the density, what is the lowest power of the density which has different coefficients for two isomeric substances?

We are now able to answer this question with a fair amount of exactness in the case of the two isomers, normal pentane and

isopentane. If, as seems most probable, there is a difference between the $\frac{1}{av^2}$ for normal pentane and the $\frac{1}{av^2}$ for isopentane, even at infinitely large volumes, this shows that the coefficients of the second power of the density in the expansion of p must be different for the two substances. On the other hand, if there is no difference between the $\frac{1}{av^2}$ for normal pentane and the $\frac{1}{av^2}$ for isopentane at infinitely large volumes, then the coefficients of the second power of the density in the expansion for p must be the same, since τ has already been shown to have the same value for the two substances at infinite volumes; and the lowest power of the density which has different coefficients for the two isomers must be the third.

It was thought advisable to test these conclusions by a different method. In a former paper by one of the authors (Phil. Mag. xliv. p. 80; see also Phil. Mag. xlv. p. 105) it was shown that in the case of isopentane we might reproduce the original observations very closely by putting

$$p=\frac{\mathrm{RT}}{v}\left\{1+\frac{e}{v+k-gv^{-2}}\right\}-\frac{l}{v(v+k)},$$

where R, e, k, g, and l are constants. If we assume that this formula holds also for normal pentane, and if it be true that the difference of pressure between normal pentane and isopentane at the same temperature and volume varies ultimately as the third power of the density, then we should be able to reproduce the experimental data for normal pentane by means of the above formula, giving to R, e, and l the values already found for isopentane. We may accordingly take $\mathrm{R}=863{\cdot}56$, $e=7{\cdot}473$, $l=5420800$, and we still have the constants k and g left at our disposal to meet the requirements of the normal pentane data. On examining the observations given in Series I. of this paper we find that we can conveniently put $k=3{\cdot}135$, $g=6{\cdot}695$, and we have to test how far the formula with these constants reproduces the experimental results given in Series II., III., and IV. In order to institute an effective comparison between theory and observation a diagram was made in which pv was plotted against $v^{-\frac{1}{2}}$;

the calculated isothermals were drawn as continuous lines, while the experimental values were put in as dots. An examination of the diagram shows that a fair concordance between calculation and experiment has been secured; but the agreement is not so good as could be wished. Deviations amounting to 1 per cent. are not uncommon, and in places they approach 2 per cent. If we have regard to the differences which often occur in inquiries of this kind between the results of independent observers, we might conclude that the above deviations are unimportant, and that R, e, and l were really the same for the two pentanes as supposed. But it seems more likely that the deviations are too large to be neglected; hence, the most probable inference is that the l for normal pentane is not the same as the l for isopentane, thereby confirming our former conclusion.

Discussion.

Dr. Lehfeldt asked whether the authors had observed any other singularity or discontinuity at vol. 3·4. He also asked whether the authors were satisfied with ordinary squared-paper in plotting their curves. It ought to be possible to design a machine for doing the work mechanically to an accuracy of $\frac{1}{50}$ of a millim.

Mr. Appleyard said the fractionating apparatus devised by Dr. Young was a great improvement on older forms; it ensured that there should always be sufficient, and yet not too much, liquid at each valve-trap. He hoped that details of the tube, in the latest form, would be included in the paper. In the separation of such a mixture as chloroform and alcohol, the common method by water-extraction was imperfect; it was not desirable always to convert the mixture wholly into chloroform. Ordinary fractionating-tubes yielded an impure distillate in this case. Perhaps the difficulty was inherent for these two liquids. Dr. Young's apparatus would put the question beyond doubt.

Dr. Young, in reply, said that the only objection to curve-tracers was their cost. The curves he had obtained from his experimental results were all isothermals; he did not think isobars would indicate anything such as Dr. Lehfeldt had

suggested. With regard to such mixtures as chloroform and alcohol, the chances of separation were difficult to predict. A distinction might however be drawn between liquids partially miscible, and liquids miscible in all proportions. Hexane (b.p. 69° C.) and benzene (b.p. 80° C.) for instance, were both hydrocarbons miscible in all proportions, and it might be thought possible to separate them by a fractionating apparatus; but experiment shows they cannot so be separated. Similarly, alcohol and chloroform are miscible in all proportions, and it rests with experiment to decide whether, on distillation, they behave "normally," or whether, like benzene and hexane, they behave as though they are only partially miscible. If they behave "normally," they *can* be separated by fractional distillation, but if they behave like partially miscible liquids, separation by that means is impossible.

XXXIV. *Repetition of an Experiment on a Magneto-optic Phenomenon discovered by Prof. Righi.* *By* Prof. Silvanus Thompson, *D.Sc., F.R.S.**

[Abstract.]

The experiment was originally described in *Atti R. Accad. Lincei, Roma*, ser. 5, vii. 1898.

A substance absorptive of light is submitted to a powerful magnetic field between the pole-pieces of an electromagnet; the pole-pieces being drilled so that a beam of light from an arc-lamp can traverse the gap *along* the magnetic lines. A polarizing prism is placed between the arc-lamp and the electromagnet, and after having passed through the magnetizing apparatus the beam thus polarized is examined by an analyzer. The analyzer must be turned to "extinction" before the magnetizing current is turned on. If this is done, brightness is restored at the analyzer as soon as the magnetic field is established. The substance absorbing the light in the gap may be nitric-oxide fumes, or an ordinary spirit-lamp sodium-flame. The second effect to be noticed is

* Read October 28, 1898.

that when the emergent beam is examined there is a splitting of the lines. Righi explains this by supposing that when light of frequency n is brought into a magnetic field and passed *along* the lines of the field, it is split up into two sets of circular waves, a right-handed and a left-handed set, one of which sets is accelerated and the other retarded. There are now two frequencies, n_1, n_2, one a little higher and the other a little lower than n. But since the analyzer is adjusted to extinguish n, there is brightness for n_1 and n_2. Normally, nitric oxide absorbs green, and red is observed; but when the magnetic field is set up, blue-green light is seen at the analyzer; for there are now two different kinds of light being absorbed, one of higher and one of lower frequency than the normal, and what is observed is the complementary spectrum. Again, if a tube of sodium is warmed to a point far short of that which would cause it to emit visible rays, and the vapour is placed in the magnet gap, at the moment when the magnetic field is set up the D lines, each doubled as in Zeeman's experiment, become visible in the observing spectroscope, *i. e.*, the emission spectrum is obtained of a substance which is not actually emitting light.

Discussion.

Mr. Blakesley said that no doubt the analyzer was used at the position of extinction for convenience merely. In other positions the eye would be overwhelmed with light.

XXXV. *A Temperature Tell-tale.*

By Rollo Appleyard.

[Abstract.]

This instrument is intended to be used in connection with vats, and for other purposes where an alarm is to be sounded by making electric contact when temperature rises or falls beyond certain limits. A J-shaped glass tube has its short limb sealed and its long limb open. Water or other suitable liquid is poured in, completely filling the short

* Read January 27, 1899.

limb. Mercury is then made to displace nearly all the water in the short limb; the surplus water in the long limb is removed by a pipette, and the mercury is adjusted to a convenient level. Two platinum contact-wires are sealed into the glass at a short distance above the free surface of mercury in the long limb. The tube may be half an inch in diameter, with a long limb of 5 inches and a short limb of $2\frac{1}{2}$ inches. The quantity of mercury in the tube is generally arranged so that, at temperatures below the boiling-point of the contained liquid, the mercury level is lowest in the long limb. In this case, if the temperature is raised to the boiling-point of the contained liquid, the mercury assumes approximately a common level in both limbs, for at the boiling-point of the liquid, under these conditions, the vapour-pressure is equal to the barometric pressure. Hence the liquid and the mercury are not spurted out.

Discussion.

Mr. Whipple said that when working with an ordinary "thermometer"-tube the contacts were inefficient, owing to oxidation. Moreover, the mercury column broke up, and in some cases mercury clung to the contact-wire. He asked if these difficulties occurred in Mr. Appleyard's apparatus.

Mr. Appleyard, in reply, explained that the change of level in the "tell-tale" was a sudden rise of about an inch of mercury, in a tube about half an inch in diameter. This rise was able completely to envelope the contact-wires, surface oxidation could not affect the working, and there could be no such thing as failure of contact. Moreover, the tube was too wide for mercury to be held up by capillarity. The large area of contact enabled the instrument to be used for strong currents. The cost was small, and the only adjustment consisted in choosing a liquid of suitable boiling-point; for the platinum wires could be sealed in anywhere in the long limb—about $2\frac{1}{2}$ inches from the bottom was a good position for them. The sudden rise occurred when the temperature was one or two degrees above the boiling-point of the contained liquid.

XXXVI. *The Volume-changes which accompany Solution.* *By* T. H. LITTLEWOOD, *M.A.**

[Abstract.]

THE apparatus for measuring the contraction observed when solids are dissolved in a liquid consists of two glass bulbs arranged one above the other, so that liquid can pass from the upper one to the lower one through a stopcock, and from the lower one upwards into the neck of the upper one through a second stopcock. This neck, which forms the top of the upper bulb, is fitted with an indiarubber stopper. The lower bulb is tubulured and provided with a glass stopper. A horizontal capillary-tube is fitted into the indiarubber stopper, so that volume-changes can be determined, after the manner of that used in Bunsen's calorimeter. The weighed solid is introduced at the tubulure. The measured amount of water is poured in at the neck. Paraffin oil is now poured in at the tubulure, so that the apparatus is completely filled—the lower bulb with the solid and the oil, the upper bulb with the water. The apparatus is then exhausted, and finally it is placed in a tank of water at constant temperature. When the stopcock between the bulbs is opened, solution begins, and the resulting contraction is measured. For small amounts of salt dissolved in constant volume of liquid, the contraction is very nearly proportional to the amount of salt. For larger amounts, the contraction is greater in proportion than the added salt. If a strong solution is gradually diluted, then, for equal amounts of water added, the contraction becomes smaller for successive amounts of added water.

The author finds Ostwald's theory of osmotic pressure insufficient to express his results, and proposes to add a term depending on the mutual attractions of the dissolved particles, somewhat after the manner of Van der Waals's theory of gases. This, when worked out, expresses the contraction as a logarithmic function of the volumes, which agrees with the observed results better.

DISCUSSION.

Dr. LEHFELDT thought the indiarubber stopper was a weak point in the apparatus. With regard to the theory of

* Read January 27, 1899.

the contraction, Tammann (*Zeitschr. für phys. Chem.*, 1894) had given an expression which was rather more intelligible. Tammann found that the effect could be regarded as equivalent to a change of pressure, and, by attributing this quality to the solution, the characteristic surface becomes the same as that for water. The volume of the solution would thus follow similar changes to those that water undergoes with increasing pressures.

Prof. EWING said the experiments reminded him of the very first piece of research work he had done in physics, which was twenty-five years ago, on the same problem, with Prof. McGregor. An examination of the electrical properties of solutions of certain salts led to an investigation of their changes of density and volume. In some cases the contraction observed was so great that the volume of the solution was less than the original volume of the water to which the salt had been added. They made some measurements, but the apparatus they had used was very rough compared to that described by Mr. Littlewood.

Mr. WATSON asked whether Mr. Littlewood had made any simultaneous density measurements. There were some solutions for which the contractions and corresponding densities had been worked out; and by examining successive dilutions of strong solutions and the corresponding densities, a check might be made as to the numerical results obtained by Mr. Littlewood's apparatus.

Dr. CHREE suggested that, in place of the logarithmic expression, the experimental results might possibly be better represented by a few terms of a series involving integral powers of the difference of volume, each term being multiplied by a proper constant, and asked whether the author had tried a series of this kind.

Mr. LITTLEWOOD, in reply, said that the indiarubber stopper possessed advantages in regulating the height of the liquid in the capillary tube. It did not introduce sensible error, for it was possible to work with it to a centigram of mercury in the capillary. On the other hand, temperature-changes of one-tenth of a degree involved 1 centigram more or less of mercury in the tube. Bunsen, using an indiarubber stopper, obtained very accurate results in calorimetry.

XXXVII. *Presidential Address delivered by* Prof. OLIVER LODGE, *LL.D., D.Sc., F.R.S., at the Annual General Meeting on February* 10*th*, 1899.

PART I.—*Preliminary Portion.*

IN taking the Chair of this Society for one year, notwithstanding that I am a country member and therefore shall not be able always to be present, I am I daresay traversing the views of what is considered best for the Society by a minority of members. I have no knowledge that it is so, but it can hardly be otherwise. I hesitated myself for some time, but ultimately consented to be put in nomination by the Council; and I can promise that I shall try to be present as often as my duties permit.

I have been a member of the Society from its earliest infancy, for though not technically an "original member," nor one admitted in the first year, that is a mere accident of not having presented myself for admittance by the Chair; perhaps I had not paid my subscription, but I used to attend all the meetings, and my memory goes back to the preliminary discussions that preceded the birth of the Society, between Prof. Foster, Prof. Adams, Dr. Gladstone, Dr. Atkinson, and Dr. Guthrie, especially Dr. Guthrie, to whom the idea of the Society was, I believe, largely due.

In those days we used to meet at South Kensington on Saturday afternoons; and my experience of recent years is altogether too remote and detached to make me aware whether the change, from the hospitable laboratory at South Kensington to the no doubt equally hospitable but less *physically* inspiring quarters of Burlington House, has been an unmixed advantage. If it is an unmixed advantage, the Society has been fortunate.

Our death-roll for the past year contains the names of two original members whose figures were familiar at the

meetings in those early days: Henry Perigal and Latimer Clark. It also contains the names of Sir J. N. Douglass, so well known in connexion with lighthouse work; of Dr. Obach, the skilled electrician and designer of Messrs. Siemens Bros.; of the Rev. Bartholomew Price, known not only as a mathematician but also as a man of public spirit and enterprize in the University of Oxford, who had I believe more to do with the early financial success of the Clarendon Press than is generally known; of Dr. J. E. Myers, and alas of Dr. John Hopkinson.

It is unfitting that this year's deliverance from the Chair of any Society of which John Hopkinson was a member should fail to contain some reference to his untimely loss; yet there is nothing for me to say but what you already feel, and no words of mine are needed. I cannot say that I knew him in his youth, but I heard of him. He and I as self-taught students—at least, I was at that time self-taught, and he could hardly have found a better teacher than himself—once figured together in the so-called honours list of the South Kensington May examinations in Electricity and Magnetism or else in Sound Light and Heat, or in both. Needless to say, his name stood above mine. He passed, for fun, in nearly the whole range of South Kensington subjects; I took a good many of them, but had nothing like either his comprehensive range or his educational advantages. And then he went and wrested the Senior Wranglership from that exceptionally brilliant mathematical genius, J. W. L. Glaisher. That was a surprising achievement for a Whitworth Scholar and practical Engineer. May I be allowed to congratulate Mrs. Hopkinson and her family not only on a peculiarly noble donation, but also on its special application to the welding together the names of Cambridge, of Engineering, and of Hopkinson.

It is remarkable, as Mr. Arthur Balfour said in a speech the other day (at the opening of the Battersea Institute, I think), it is remarkable how in modern times the pursuit of science tends in the direction of rapid and immediate application.

In Newton's time pure science was altogether aloof from

practice: I do not suppose that the gravitational theory of astronomy had any effect on His Majesty's coins at the Mint, nor did the prismatic sifting of white-light lead at that time to any chemical applications of spectrum analysis. Even as late as Faraday's day the induction of currents by motion and magnetism exercised no appreciable effect on commerce and industry.

Nowadays, Fourier's theorem is applied by Lord Kelvin to work an Atlantic cable; Hopkinson's researches on the magnetic circuit (an idea that has revolutionized the practical treatment of magnetism) result at once in improvements in the dynamo; and Hertz's experimental detection of Maxwell's waves seems likely to result before long in a new system of wireless telegraphy.

What is the cause of this keen interest in practical applications felt in varying degrees by nearly all modern physicists, felt most keenly and consistently I think, of those I know, by your past-President Prof. Ayrton? Is it the example and inspiration of Lord Kelvin, or is it that the human race generally is beginning to be better educated than it used to be, to take more kindly to the result of scientific researches, to be more eager to learn about them and, if possible, to apply them; partly, no doubt, because it has gradually discovered that by judicious treatment it can convert them into substantial commodity?

Surely, on the whole, with some drawbacks, the result from the point of view of science is good. If science were really remote from all human use or interest, as some scoffers still endeavour to maintain it to be, would there not be a fear lest gradually the human race should get tired of it, should cease to encourage even by approval or applause those whose instinct or mission it might be to develop it? So that gradually a blight might settle down, and advanced scientific knowledge become as extinct as a dead language or a fossil genus. A few scholars here and there would take an interest in it, as they did in the Middle Ages, and it might perhaps for a time be instilled into youth at the point of a cane, as the dead languages not so long ago used to be; but unless a branch of learning enters in some way into the life and well-

being of a community, unless it has some real interest—a literary interest it may be, a commercial interest perhaps, a living human interest of some kind—there is fear that however artificially bolstered up it may be for a time, by those whose stock in trade it is and who know little else, it will ultimately be suffered to go by the board and become to all practical intents and purposes dead, buried, and extinct.

I said just now that I supposed the public as a whole were gradually getting better educated, but how miserably slow is the process and how lamentable the present result. I have no feud with orthodox secondary schoolmasters, indeed some of them are my very good friends, but is it not painful to be unable to speak to the average so-called educated Englishman (except a few here and there of course) on any scientific discovery without at once being met by a hopeless wall of ignorance.

Is it so in other countries? I hardly think so; I do not think it is so marked even among Scotchmen, at any rate not among Scotchmen who have had their Natural Philosophy year at the University. A year for training in the whole of science—it is not much. If scientific men were ignorant of letters in the same proportion they would hardly be able to read, except in words of one syllable, nor write, except with the laborious contortions of the village urchin; a year extended on the subjects of all the Arts would not carry one very far. And as to sums, how many men there are whose mathematical equipment is limited to the practice of compound addition and subtraction, and who shy if they see an algebraic symbol, even in a newspaper,—indeed, when they occur in newspapers they are often of a kind that deserve to be shyed at.

They learn languages, some people seem to enjoy learning foreign languages—though the multiplicity of them is an artificial arrangement which an international convention might alter if it chose,—but the exact and beautiful language in which a great part of science is and must be written is to the majority of men unknown. Are teachers wholly free from blame in respect of the dislike which many boys feel

for the most elementary mathematics, for problems which may be made exhilarating when properly presented? I believe they are not. If the teacher dislikes mathematics, the boys will hate it, and if they hate it when young they will love it but little better when they grow up and proceed to be teachers themselves. There is a law of geometrical progression or compound interest here, and it seems to have been at work for some time. I must not however be understood here to be speaking of really cultivated intelligences or authorities in any subject. A thorough training in any branch of knowledge may so stimulate and enlarge the faculties as to fit them for intelligent appreciation and criticism of many other branches. Some men of letters I know whose minds are copiously and essentially scientific, they have not the detailed knowledge of the professed worker in science—that would be impossible and unnecessary—but they appreciate and assimilate all the main doctrines of science, and their broad criticisms and suggestions are often of value to the special worker. Any educational strictures on which I have ventured have reference not to scholars but to average people; at the same time I believe many scholars feel their deficiency in mathematics, and I believe it to be the result of bad teaching.

This is not my address, it is a preliminary excursus, my address has reference to a physical problem which has interested me, and about which I thought it might be interesting if I said something. It is customary to review some topic in an address, not to adduce anything specially new, but to treat of or attempt to clarify something already known. Usually it is some experimental fact which is thus treated, sometimes it is a point of theory; it is the latter to which I wish to ask your attention.

As to experiments, if I quickly review some of those that strike me in the past year—there is further progress to be reported concerning the developments of Prof. Zeeman's great discovery, the most important being (1) the new manifestations of it, or of a closely allied phenomenon, discovered by Professor Righi, to which Prof. S. P. Thompson called the attention of Section A at Bristol last September; and

(2) some extension of the resolution of complex lines, discriminating between the effect as exhibited by different lines, a discrimination which seems likely to lead to an extension of the powers of spectrum analysis by the application of a magnetic field to the source of light, as studied with apparatus of splendid dispersive power by Prof. Preston, and with his own ingenious method by Prof. Michelson. There is a notable continuation of the remarkably powerful and beautiful researches into atomic properties carried on at Cambridge by Prof. J. J. Thomson ; about which a great deal more might be said. And then there is the prediction made in the columns of 'Nature' by Prof. G. F. FitzGerald, but not yet verified, that circularly polarized light sent through an absorbing medium would constitute a magnet.

Many have been the attempts, from Faraday downwards, to excite magnetization by means of light, to detect a real and not an imaginary "magnetization of a ray of light," but all the experimenters hitherto have failed to realise the necessity of an absorbent medium. The effect is bound to be small, and of course it may not be there after all, but I understand it is being looked for, and it has at any rate been on rational grounds predicted.

Dr. Barton has continued experiments on the damping of waves on wires, and has not yet reconciled a discrepancy between theory and experiment ; being able, as I understand him, to account for only 5½ per cent. of the outstanding discrepancy. I have not myself looked into the matter, but I understand that a note has been communicated by Mr. Heaviside. I hope it will be published in full.

The subject of Terrestrial Magnetism, in the hands of Prof. Schuster on the theoretical side and of Prof. Rücker on the practical side, has recently entered on an enlarged existence and seems to be taking up a position almost of an independent science, so that when the multifurcation or explosion of Section A occurs, as sooner or later I fear it must, the fragment entitled Meteorology and Terrestrial Magnetism will perhaps not be one of the smallest.

The publication called 'Science Abstracts' has continued to be welcomed by workers all over the country, and the abstracts sent by Mr. Fournier d'Albe to the 'Electrician' are also useful and admirably done. I have no knowledge myself of the mode of control of the publication of our 'Science Abstracts,' but I confess to a personal preference for the older form, the one with a less startling cover, and with a list of authors' names, instead of unnecessary advertisements of well-known firms, on its back page. Undoubtedly the inauguration of these 'Abstracts' supplies a long-felt want, and even if one does not always find time to read them it is a comfort to feel that there they are, to be read in some less pressing season; and one need not envy the German worker his 'Beiblätter' as one used to.

But there is one most important event that has occurred during the year, perhaps to us conjointly as physicists the most important of all, an event of which the science of Physics will feel the effects—I trust the wholly favourable effects—for centuries to come: I mean the decision of the Government to begin on a small scale the inauguration of a National Physical Laboratory. If its effects are ever in any respects unfavourable to true science, a heavy indictment will lie against its future governing body, and against its Superintendent or Physicist Royal.

I wish to congratulate Sir Douglas Galton, as well as to some extent myself, on the result, the speedy result, of our urging of the subject, within the present decade, on the British Association; and as a Physical Society I think we owe a debt of gratitude to the Committee appointed by the Treasury to examine into the question, whose members gave themselves an immense amount of trouble, travelling to Berlin and elsewhere to make themselves acquainted with foreign ideals, and altogether going into the subject with extreme thoroughness, scientific judgment, and public spirit. The chairman of the committee was Lord Rayleigh, but during his absence in India the acting chairman was constantly Professor Rücker; Mr. Chalmers represented the Treasury with urbane frugality; and to Prof. Rücker and

Mr. Chalmers, as well as to the other members of the committee, I am sure you will agree with me our thanks are justly due.

The record of their labours and of the voluminous evidence which they obtained, to which evidence indeed several of us here contributed, is embodied in a 'blue-book' of more than usual interest and instruction. I commend the perusal of it not only to professed physicists, but to all scientific teachers and their advanced students, that they may learn what accuracy is, and what it is practically wanted for.

It is not for me as your President to touch at any length upon outside or political topics, but briefly I venture to think that the year 1898 has been a year of no ordinary good fortune, at least to the Anglo-Saxon race. A year, and a Government, which has witnessed and assisted not only the beginning of a National Physical Laboratory, but such great outside events as the Imperial expansion and cooperation of America, (for America, and not the United States, it will now have to be called), the liberation of Crete, and the restoration of the civilization of Upper Egypt, is a year upon which we can look back with satisfaction, and is a Government which, irrespective of party, we may legitimately congratulate, and even, though that I admit is unusual, thank.

Would that we could add to its laurels the commencement on wise lines of a real and comprehensive, dignified and progressive, genuine University of London.

PART II.—*On Opacity.*

MY attention has recently been called to the subject of the transmission of electromagnetic waves by conducting dielectrics—in other words, to the opacity of imperfectly conducting material to light. The question arose when an attempt was being made to signal inductively through a stratum of earth or sea, how far the intervening layers of moderately conducting material were able to act as a screen; the question also arises in the transmission of Hertz waves through partial conductors, and again in the transparency of gold-leaf and other homogeneous substances to light.

The earliest treatment of such subjects is due of course to Clerk Maxwell thirty-four years ago, when, with unexampled genius, he laid down the fundamental laws for the propagation of electric waves in simple dielectrics, in crystalline media, and in conducting media. He also realised there was some strong analogy between the transmission of such waves through space and the transmission of pulses of current along a telegraph-wire. But naturally at that early date not every detail of the investigation was equally satisfactory and complete.

Since that time, and using Maxwell as a basis, several mathematicians have developed the theory further, and no one with more comprehensive thoroughness than Mr. Oliver Heaviside, who has gone into these matters with extraordinarily clear and far vision. I may take the opportunity of calling or recalling to the notice of the Society, as well as of myself, some of the simpler developments of Mr. Heaviside's theory and manner of unifying phenomena and processes at first sight apparently different; but first I will deal with the better-known aspects of the subject.

Maxwell deals with the relation between conductivity and opacity in his Art. 798 and on practically to the end of that famous chapter xx. ('Electromagnetic Theory of Light'). He discriminates, though not very explicitly or obtrusively, between the two extreme cases, (1) when inductive capacity

or electric inductivity is the dominant feature of the medium—when, for instance, it is a slightly conducting dielectric, and (2) the other extreme case, when conductivity is the predominant feature.

The equation for the second case, that of predominant conductivity, is

$$\frac{d^2F}{dx^2} = \frac{4\pi\mu}{\sigma}\frac{dF}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

F being practically any vector representing the amplitude of the disturbance; for since we need not trouble ourselves with geometrical considerations such as the oblique incidence of waves on a boundary &c., we are at liberty to write the ∇ merely as d/dx, taking the beam parallel and the incidence normal.

No examples are given by Maxwell of the solution of this equation, because it is obviously analogous to the ordinary heat diffusion fully treated by Fourier.

Suffice it for us to say that, taking F at the origin as represented by a simple harmonic disturbance $F_0 = e^{ipt}$, the solution of equation (1)

$$\frac{d^2F}{dx^2} = \frac{4\pi\mu ip}{\sigma}F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

is

$$F = F_0 e^{-Qx} = e^{-Qx+ipt},$$

where

$$Q = \sqrt{\left(\frac{4\pi\mu ip}{\sigma}\right)} = \sqrt{\frac{2\pi\mu p}{\sigma}}\,.\,(1+i);$$

wherefore

$$F = e^{-\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x} \cos\left(pt - \left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x\right), \quad . \quad . \quad . \quad (2)$$

an equation which exhibits no true elastic wave propagation at a definite velocity, but a trailing and distorted progress, with every harmonic constituent going at a different pace, and dying out at a different rate; in other words, the *diffusion* so well known in the case of the variable stage of heat-conduction through a slab.

In such conduction the gain of heat by any element whose heat capacity is $c\rho dx$ is proportional to the difference of the

temperature gradient at its fore and aft surfaces, so that

$$cp\,dx\,\frac{d\theta}{dt}=d\,.\,k\,\frac{d\theta}{dx},$$

or, what is the same thing,

$$\frac{d^2\theta}{dx^2}=\frac{cp}{k}\frac{d\theta}{dt},$$

the same as the equation (1) above; wherefore the constant cp/k, the *reciprocal* of the thermometric conductivity, takes the place of $4\pi\mu/\sigma$, that is, of electric conductivity; otherwise the heat solution is the same as (2). The 4π has come in from an unfortunate convention, but it is remarkable that the conductivity term is inverted. The reason of the inversion of this constant is that, whereas the *substance* conveys the heat waves, and by its conductivity aids their advance, the *æther* conveys the electric waves, and the substance only screens and opposes, reflects, or dissipates them.

This is the case applied to sea-water and low frequency by Mr. Whitehead in a paper which he gave to this Society in June 1897, being prompted thereto by the difficulty which Mr. Evershed and the Post Office had found in some trials of induction signalling at the Goodwin Sands between a coil round a ship at the surface and another coil submerged at a depth of 10 or 12 fathoms. It was suspected that the conductivity of the water mopped up a considerable proportion of the induced currents, and Mr. Whitehead's calculation tended, or was held to tend, to support that conclusion.

To the discussion Mr. Heaviside communicated what was apparently, as reported, a brief statement; but I learn that in reality it was a carefully written note of three pages, of which recently he has been good enough to lend me a copy. In that note he calls attention to a theory of the whole subject which in 1887 he had worked out and printed in his collected 'Electrical Papers,' but which has very likely been overlooked. It seems to me a pity that a note by Mr. Heaviside should have been so abridged in the reported discussion as to be practically useless; and I am permitted to quote it here as an appendix (p. 413).

Meanwhile, taking the diffusion case as applicable to sea-water with moderately low acoustic frequency, we see that the

induction effect decreases geometrically with the thickness of the oceanic layer, and that the logarithmic decrement of the amplitude of the oscillation is $\sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)}$, where σ is the specific resistance of sea-water and $p/2\pi$ is the frequency.

Mr. Evershed has measured σ and found it 2×10^{10} C.G.S., that is to say $2\times10^{10}\,\mu$ square centim. per second; so putting in this value and taking a frequency of 16 per second, the amplitude is reduced to $1/e$th of what its value would have been at the same distance in a perfect insulator, by a depth

$$\sqrt{\frac{\sigma}{2\pi\mu p}}=\sqrt{\left(\frac{2\times10^{10}\mu}{2\pi\mu\times2\pi\times16}\right)}=\sqrt{\frac{10^{10}}{320}}=\frac{10^{5}}{18}\text{ centim.}$$
$$=55\text{ metres.}$$

Four or five times this thickness of intervening sea would reduce the result at the 16 frequency to insignificance (each 55-metre-layer reducing the energy to $\frac{1}{7}$ of what entered it); but if the frequency were, say, 400 per second instead of 16 it would be five times more damped, and the damping thickness (the depth reducing the amplitude in the ratio $e:1$) would in that case be only eleven metres.

It is clear that in a sea 10 fathoms (or say 20 metres) deep the failure to inductively operate a "call" responding to a frequency of 16 per second was *not* due to the screening effect of sea-water *.

Maxwell, however, is more interested in the propagation of actual light, that is to say, in waves whose frequency is about 5×10^{14} per second; and for that he evidently does not consider that the simple diffusion theory is suitable. It certainly is not applicable to light passing through so feeble a conductor as salt water. He attends mainly therefore to the other and more interesting case, where electric inductive-capacity predominates over the damping effect of conductivity, and where true waves therefore advance with an

* I learn that the ship supporting the secondary cable was of metal, and that the primary or submerged cable was sheathed in uninsulated metal, viz. in iron, which would no doubt be practically short-circuited by the sea-water. Opacity of the medium is in that case a superfluous explanation of the failure, since a closed secondary existed close to both sending and receiving circuit.

approximately definite velocity

$$v = \frac{1}{\sqrt{\mu K}};$$

though it is to be noted that the slight sorting out of waves of different frequency, called dispersion, is an approximation to the case of pure diffusion where the speed is as the square root of the frequency, and is accompanied, moreover, as it ought to be, by a certain amount of differential or selective absorption.

To treat the case of waves in a conductor, the same damping term as before has to be added to the ordinary wave equation, and so we have

$$\frac{d^2F}{dx^2} = \mu K \frac{d^2F}{dt^2} + \frac{4\pi\mu}{\sigma}\frac{dF}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking $F_0 = e^{ipt}$ again, it may be written

$$\frac{d^2F}{dx^2} = \left(-\mu K p^2 + \frac{4\pi\mu i p}{\sigma}\right)F, \quad . \quad . \quad . \quad . \quad (3')$$

the same form as equation (1′); so the solution is again

$$F = e^{-Qx + ipt},$$

with Q^2 equal to the coefficient of F in (3′). Maxwell, however, does not happen to extract the square root of this quantity, but, assuming the answer to be of the form (for a simply harmonic disturbance) [modifying his letters, vol. ii. § 798]

$$e^{-rx}\cos(pt - qx),$$

he differentiates and equates coefficients, thus getting

$$q^2 - r^2 = \mu K p^2, \qquad 2rq = \frac{4\pi\mu p}{\sigma},$$

as the conditions enabling it to satisfy the differential equation. This of course gives for the logarithmic decrement, or coefficient of absorption,

$$r = \frac{2\pi\mu}{\sigma} \cdot \frac{p}{q},$$

p/q being precisely the velocity of propagation of the train of waves. Though not exactly equal to $1/\sqrt{\mu K}$, the true velocity of wave propagation, except as a first approximation, in an absorbing medium, yet practically this velocity p/q or λ/T

is independent of the frequency except in strongly absorbent substances where there are dispersional complications; and so the damping is, in simple cases, practically independent of the frequency too.

With this simple velocity in mind Maxwell proceeds to apply his theory numerically to gold-leaf, calculating its theoretical transparency, and finding, as every one knows, that it comes out discordant with experiment, being out of all comparison * smaller than what experiment gives.

But then it is somewhat surprising to find gold treated as a substance in which conductivity does not predominate over specific inductive capacity.

The differential equation is quite general and applies to any substance, and since the solution given is a true solution, it too must apply to any substance when properly interpreted; but writing it in the form just given does not suggest the full and complete solution. It seems to apply only to slightly damped waves, and indeed, Maxwell seems to consider it desirable to rewrite the original equation with omission of K, for the purpose of dealing with good conductors.

By a slip, however, he treats gold for the moment as if it belonged to the category of poor conductors, and as if absorption in a thickness such as gold-leaf could be treated as a moderate damping of otherwise progressive waves.

The slip was naturally due to a consideration of the extreme frequency of light vibrations; but attention to the more complete expression for the solution of the same differential equation, given in 1887 by Mr. Heaviside and quoted in the note to this Society above referred to, puts the matter in a proper position. Referring to his 'Electrical Papers,' vol. ii. p. 422, he writes down the general value of the coefficient of absorption as follows (translating into our notation)

$$r=\frac{p}{v\sqrt{2}}\left\{\left[1+\left(\frac{4\pi}{\sigma p\mathrm{K}}\right)^2\right]^{\frac{1}{2}}-1\right\}^{\frac{1}{2}}$$

without regard to whether the conductivity of the medium is large or small; where v is the undamped or true velocity of wave propagation in the medium $(\mu\mathrm{K})^{-\frac{1}{2}}$.

* The fraction representing the calculated transmission by a film half a wave thick has two thousand digits in its denominator: see below, top of p. 370.

Of course Maxwell could have got this expression in an instant by extracting the square root of the quantity Q, the coefficient of F in equation (3′) written above. I do not suppose that there is anything of the slightest interest from the mathematician's point of view, the interest lies in the physical application; but as this is not a mathematical Society it is permissible, and I believe proper, to indicate steps for the working out of the general solution of equation (3) by extracting the square root of the complex quantity Q.

The equation is

$$\frac{d^2F}{dx^2}+\left(\mu K p^2-\frac{4\pi\mu i p}{\sigma}\right)F=0,$$

and the solution is

$$F=e^{-Qx+ipt},$$

where

$$Q=\sqrt{-\mu K p^2+\frac{4\pi\mu i p}{\sigma}}\quad=\alpha+i\beta \text{ say.}$$

Squaring we get, just as Maxwell did,

$$\alpha^2-\beta^2=-\mu K p^2,\quad 2\alpha\beta=+\frac{4\pi\mu p}{\sigma}.$$

Squaring again and adding

$$(\alpha^2+\beta^2)^2=(\alpha^2-\beta^2)^2+4\alpha^2\beta^2=\mu^2K^2p^4+\frac{16\pi^2\mu^2p^2}{\sigma^2}$$

$$\therefore\quad \alpha^2+\beta^2=\mu K p^2\left\{1+\left(\frac{4\pi}{Kp\sigma}\right)^2\right\}^{\frac{1}{2}},$$

wherefore

$$2\beta^2=\mu K p^2\left\{\sqrt{\left(1+\left(\frac{4\pi}{Kp\sigma}\right)^2\right)}+1\right\}, \quad . \quad . \quad (4)$$

and $2\alpha^2$ = the same with the last sign negative,

or
$$\alpha=p\sqrt{(\tfrac{1}{2}\mu K)}\Big\}\left[1+\left(\frac{4\pi}{\sigma p K}\right)^2\Big\}^{\frac{1}{2}}-1\right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad (5)$$

which is the logarithmic decrement of the oscillation per unit of distance, or the reciprocal of the thickness which reduces the amplitude in the ratio $1:e$ (or the energy to $\frac{1}{e}$) of the value it would have at the same place without damping.

Using these values for α and β, the radiation-vector in general, after passing through any thickness x of any medium

whose magnetic permeability and other properties are *constant*, is

$$F = F_0 e^{-\alpha x} \cos(pt - \beta x), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the speed of advance of the wave-train being p/β.

Now not only the numerical value but the form of this damping constant α depends on the magnitude of the numerical quantity $\frac{4\pi}{\sigma p K}$, which may be called the critical number*, and may also be written

$$\frac{4\pi\mu_0 v_0^2}{p\sigma K/K_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where K, the absolute specific inductive capacity of the medium, is replaced by its relative value in terms of K_0 for vacuum, and by $\frac{1}{\sqrt{K_0\mu_0}}$ = the velocity of light *in vacuo* $= v_0$.

Now for all ordinary frequencies and good conductors this critical number is large; and in that case it will be found that

$$\alpha = p\sqrt{\frac{\mu K}{2}} \cdot \sqrt{\frac{4\pi}{\sigma p K}} = \sqrt{\frac{2\pi\mu p}{\sigma}},$$

and that β is identically the same. This represents the simple diffusion case, and leads to equation (2).

On the other hand, for luminous frequency and bad conductors, the critical quantity is small, and in that case

$$\alpha = p\sqrt{(\tfrac{1}{2}\mu K)} \left\{ 1 + \tfrac{1}{2}\left(\frac{4\pi}{\sigma p K}\right)^2 - 1 \right\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2} p \sqrt{(\mu K)} \cdot \frac{4\pi}{\sigma p K} = \frac{2\pi\mu v}{\sigma},$$

while

$$\beta = p\sqrt{\mu K} = \frac{p}{v},$$

giving the solution

$$F = F_0 e^{-\frac{2\pi\mu v}{\sigma}x} \cos p\left(t - \frac{x}{v}\right). \quad . \quad \quad . \quad (8)$$

* An instructive mode of writing α and β in general is given in (11″) or (12″) below, where the above critical number is called $\tan \epsilon$:—

$$\alpha v \sqrt{\cos\epsilon} = p \sin \tfrac{1}{2}\epsilon,$$
$$\beta v \sqrt{\cos\epsilon} = p \cos \tfrac{1}{2}\epsilon.$$

This expresses the transmission of light through imperfect insulators, and is the case specially applied by Maxwell to calculations of opacity. Its form serves likewise for telegraphic signals or Hertz waves transmitted by a highly-conducting aerial wire; the damping, if any, is independent of frequency and there is true undistorted wave-propagation at velocity $v=1/\sqrt{(LS)}$; the constants belonging to unit length of the wire. The current (or potential) at any time and place is

$$C=C_0e^{-\frac{Rx}{2Lv}}\cos p(t-x\sqrt{LS}). \quad . \quad . \quad . \quad (9)$$

The other extreme case, that of diffusion, represented by equation (2), is analogous to the well-known transmission of slow signals by Atlantic cables, that is by long cables where resistance and capacity are predominant, giving the so-called KR law (only that I will write it RS),

$$C = C_0\,e^{-\sqrt{(\frac{1}{2}pRS)}x}\cos\{pt-\sqrt{(\tfrac{1}{2}pRS)}x\}; \quad . \quad . \quad (10)$$

wherefore the damping distance in a cable is

$$x_0=\sqrt{\left(\frac{2}{pRS}\right)}.$$

Thus, in comparing the cable case with the penetration of waves into a conductor and with the case of thermal conduction, the following quantities correspond:

$$\tfrac{1}{2}pRS, \qquad \frac{4\pi\mu p}{2\sigma}, \qquad \frac{pc\rho}{2k}.$$

$c\rho$ is the heat-capacity per unit volume, S is the electric capacity per unit length; k is the thermal conductivity per unit volume, 1/R is the electric conductance per unit length. So these agree exactly; but in the middle case, that of waves entering a conductor, there is a notable inversion, representing a real physical fact. $4\pi\mu$ may be called the density and may be compared with ρ or with 1 S, that is with elasticity$\div v^2$; but σ is the *resistance* per unit volume instead of the conductance. The reason of course is that whereas good conductivity helps the cable-signals or the heat along, it by no means helps the waves into the conductor. Conductivity aids their slipping *along* the boundary of a conductor, but it retards their passing *across* the boundary and entering a

conductor. As regards waves entering a conductor, the effect of conductivity is a screening effect, not a transmitting effect, and it is the bad conductor which alone has a chance of being a transparent medium.

It may be convenient to telegraphists, accustomed to think in terms of the "KR-law" and comparing equations (2) and (10), to note that the quantity $4\pi\mu\sigma$—that is, practically, the specific conductivity in electromagnetic measure (multiplied by a meaningless 4π because of an unfortunate initial convention)—takes the place of KR (*i. e.* of RS), but that otherwise the damping-out of the waves as they enter a good conductor is exactly like the damping-out of the signals as they progress through a cable; or again as electrification travels along a cotton thread, or as a temperature pulse makes its way through a slab; and yet another case, though it is different in many respects, yet has some similarities, viz. the ultimate distance the melting-point of wax travels along a bar in Ingenhousz's conductivity apparatus,—the same law of inverse square of distance for effective reach of signal holding in each case.

Now it is pointed out by Mr. Heaviside in several places in his writings that, whereas the transmission of high-frequency waves by a nearly transparent substance corresponds by analogy to the conveyance of Hertz waves along aerial wires (or along cables for that matter, if sufficiently conducting), and whereas the absorption of low-frequency waves by a conducting substance corresponds, also by analogy, to the diffusion of pulses along a telegraph-cable whose self-induction is neglected—its resistance and capacity being prominent,—the intermediate case of waves of moderate frequency in a conductor of intermediate opacity corresponds to the more general cable case where self-induction becomes important and where leakage also must be taken into account; because it is leakage conductance that is the conductance of the dielectric concerned in plane waves. This last is therefore a real, and not only an analogic, correspondence.

Writing R_1 S_1 L_1 Q_1 for the resistance, the capacity ("permittance"), the inductance, and the leakage-conductance ("leakance") respectively, per unit length, the general equations to cable-signalling are given in Mr. Heaviside's

'Electromagnetic Theory' thus :—

$$S_1\frac{dV}{dt}+Q_1V+\frac{dC}{dx}=0, \quad L_1\frac{dC}{dt}+R_1C+\frac{dV}{dx}=0;$$

or for a simple harmonic disturbance,

$$\frac{d^2V}{dx^2}=(R_1+ipL_1)(Q_1+ipS_1)V \quad . \quad . \quad . \quad (11)$$

$$=(\alpha+i\beta)^2V,$$

whose solution therefore is

$$V=V_0\,e^{-\alpha x}\cos(pt-\beta x)\,*. \quad . \quad . \quad . \quad (11')$$

There are several interesting special cases:—

The old cable theory of Lord Kelvin is obtained by omitting both Q and L; thus getting equation (2).

The transmission of Hertz waves along a perfectly-conducting insulated wire is obtained by omitting Q and R; the speed of such transmission being $1/\sqrt{(L_1S_1)}$. Resistance in the wire brings it to the form (9), where the damping depends on the ratio of the capacity constant RS to the self-induction constant L/S; because the index $R/2Lv$ equals half the square root of this ratio; but it must be remembered that R has the throttled value due to merely superficial penetration. The case is approximated to in telephony sometimes.

A remarkable case of undistorted (though attenuated) transmission through a cable (discovered by Mr. Heaviside, but not yet practically applied) is obtained by taking

$$R/L=Q/S=r;$$

the solution being then

$$V=e^{-\frac{rx}{v}}f\left(t-\frac{x}{v}\right),$$

due to $f(t)$ at $x=0$. All frequencies are thus treated alike,

* I don't know whether the following simple general expression for α and β has been recorded by anyone: writing $R/pL=\tan\epsilon$ and $Q/pS=\tan\epsilon'$,

$$\alpha \text{ or } \beta=\frac{p}{v}\cdot\frac{\sin \textit{ or } \cos\frac{1}{2}(\epsilon+\epsilon')}{(\cos\epsilon\cos\epsilon')^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11'')$$

which is shorter than (12).

and a true velocity of transmission makes its reappearance. This is what he calls his "distortionless circuit," which may yet play an important part in practice.

And lastly, the two cases which for brevity may be treated together, the case of perfect insulation, $Q=0$, on the one hand, and the case of perfect wire conduction, $R=0$, on the other. For either of these cases the general expression

$$\alpha^2 \text{ or } \beta^2 = \tfrac{1}{2}p^2 L_1 S_1 \left[\left\{ 1 + \left(\frac{R}{pL}\right)^2 \right\}^{\frac{1}{2}} \left\{ 1 + \left(\frac{Q}{pS}\right)^2 \right\}^{\frac{1}{2}} \pm \left\{ \frac{RQ}{p^2 LS} - 1 \right\} \right] \quad (12)$$

becomes exactly of the form (4) or (5) reckoned above for the general screening-effect, or opacity, of conducting media in space.

For the number which takes the place of the quantity there called the critical number, namely either R/pL or Q/pS, the other being zero, we may write $\tan\epsilon$; in which case the above is

$$\alpha^2 \text{ or } \beta^2 = \tfrac{1}{2}p^2 L_1 S_1 (\sec\epsilon \mp 1) ; \quad . \quad . \quad . \quad . \quad . \quad (12')$$

or, rewriting in a sufficiently obvious manner, with $2\pi/\lambda$ for p/v if we choose,

$$\alpha = \frac{p \sin\frac{1}{2}\epsilon}{v(\cos\epsilon)^{\frac{1}{2}}}, \qquad \beta = \frac{p\cos\frac{1}{2}\epsilon}{v(\cos\epsilon)^{\frac{1}{2}}} . \quad . \quad . \quad . \quad (12'')$$

Instead of attending to special cases, if we attend to the general cable equation (11) as it stands, we see that it is more general than the corresponding equation (3) to waves in space, because it contains the extra possibility R of wire resistance, which does not exist in free space.

Mr. Heaviside, however, prefers to unify the whole by the introduction of a hypothetical and as yet undiscovered dissipation-possibility in space, or in material bodies occupying space, which he calls magnetic conductance, and which, though supposed to be non-existent, may perhaps conceivably represent the reciprocal of some kind of hysteresis, either the electric or the magnetic variety. Calling this g, (gH^2 is to be the dissipation term corresponding with RC^2), the equation to waves in space becomes

$$\nabla^2 F = (g + ip\mu)\left(\frac{4\pi}{\sigma} + ipK\right) F, \quad . \quad . \quad . \quad (13)$$

just like the general cable case. And a curious kind of transparency, attenuation without distortion, would belong to a medium in which both conductivities coexisted in such proportion that $g : \mu = 4\pi k : K$; for g would destroy H just as k destroys E.

In the cable, F may be either current or potential, and $LSv^2 = 1$. In space, F may be either electric or magnetic intensity, and $\mu K v^2 = 1$; but observe that g takes the place not of Q but of R, while it is $4\pi/\sigma$ that takes the place of Q. Resistance in the wire and electric conductivity in space do not produce similar effects. If there is any analogue in space to wire resistance it is magnetic not electric conductivity.

The important thing is of course that the wire does not convey the energy but dissipates it, so that the dissipation by wire-resistance and the dissipation by space-hysteresis to that extent correspond. The screening effect of space-conductivity involves the very same dielectric property as that which causes leakage or imperfect insulation of the cable core.

Returning to the imaginary magnetic conductivity, let us trace what its effects would be if it existed, and try to grasp it. It effect would be to kill out the magnetism of permanent magnets in time, and generally to waste away the energy of a static magnetic field, just as resistance in wires wastes the energy of an unmaintained current and so kills out the magnetism of *its* field. I spoke above as if it were conceivable that such magnetic conductivity could actually in some degree exist, likening it to a kind of hysteresis; but hysteresis—the enclosure of a loop between a to and fro path—is a phenomenon essentially associated with *fluctuations*, and cannot exist in a steady field with everything stationary. Admitted: but then the molecules are not stationary, and the behaviour of molecules in the Zeeman and Righi phenomena, or still more strikingly in the gratuitous radiations discovered by Edmond Becquerel, and more widely recognized by others, especially by Monsieur et Madame Curie, (not really gratuitous but effected probably by conversion into high-pitched radiation of energy supplied from low-pitched sources),—the way molecules of absorbent substances behave, seems to render

possible, or at least conceivable, something like a minute magnetic conductivity in radiative or absorptive substances. Mr. Heaviside, however, never introduced it as a physical fact for which there was any experimental evidence, but as a physical possibility and especially as a mathematical auxiliary and unifier of treatment, and that is all that we need here consider it to be; but we may trace in rather more detail its effect if it did exist.

Suppose the magnetism of a magnet decayed, what would happen to its lines of force? They would gradually shrink into smaller loops and ultimately into molecular ones. The generation of a magnetic field is always the opening out of previously existing molecular magnetic loops; there is no such thing as the creation of a magnetic field, except in the sense of moving it into a fresh place or expanding it over a wider region *. So also the destruction of a magnetic field merely means the shrinkage of its lines of force (or lines of induction, I am not here discriminating between them). Now consider an electric current in a wire:—a cylindrical magnetic field surrounds it, and if the current gradually decreases in strength the magnetic energy gradually sinks into the wire as its lines slowly collapse. But observe that the *electric* energy of the field remains unchanged by this process: if the wire were electrostatically charged it would remain charged, its average potential can remain constant. Let the wire for instance be perfectly conducting, then the current needs no maintenance, the potential might be uniform (though in general there would be waves running to and fro), and both the electric and magnetic fields continue for ever, unless there is some dissipative property in space.

Two kinds of dissipative property may be imagined in matter filling space: first, and most ordinary, an electric conductivity or simple leakage, the result of which will be to equalize the potential throughout space and destroy the electric field, without necessarily affecting the magnetic field, and so without stopping the steady circulation of the current manifested by that field. The other dissipative property in space that could be imagined would be magnetic conductivity; the result of which would be to shrink all the circular lines of

* This may be disagreed with.

magnetic force slowly upon the wire, thus destroying the magnetic field, and with it (by the circuital relation) the current; but leaving the electrostatic potential and the electric field unchanged. And this imaginary effect of the medium in surrounding space is exactly the real effect caused by what is called electric resistance in the wire *.

Now for a simply progressive undistorted wave, *i. e.* one with no character of diffusion about it, but all frequencies travelling at the same quite definite speed $1/\sqrt{\mu K}$, it is essential that the electric and magnetic energies shall be equal. If both are weakened *in the same proportion*, the wave-energy is diminished, and the pulse is said to be "attenuated," but it continues otherwise uninjured and arrives "undistorted," that is, with all its features intact and at the same speed as before, but on a reduced scale in point of size.

This is the case of Mr. Heaviside's "distortionless circuit" spoken of above, and its practical realization in cables, though it would not at once mean Atlantic telephony, would mean greatly improved signalling, and probably telephony through shorter cables. In a cable the length of the Atlantic the attenuation would be excessive, unless the absence of distortion were secured by increasing rather the wire-conductance than the dielectric leakage; but, unless excessive, simple attenuation does no serious harm. Articulation depends on the features of the wave, and the preservation of the features demands, by Fourier's analysis, the transmission of every frequency at the same rate.

But now suppose any cause diminishes one of the two fields without diminishing the other: for instance, let the electric field be weakened by leakage alone, or let the magnetic field be weakened by wire-resistance alone, then what happens? The preservation of E and the diminution of H, to take the latter—the ordinary—case, may be regarded as a superposition on the advancing wave of a gradually growing reverse field of intensity δH; and, by the relation $E = \mu v H$,

* There is this difference, that in the real case the heat of dissipation appears locally in the wire, whereas in the imaginary case it appears throughout the magnetically conducting medium; but I apprehend that in the imaginary case the lines would still shrink, by reason of molecular loops being pinched off them.

this reversed field, for whatever it is worth, must mean a gradually growing wave travelling in the reverse direction.

The ordinary wave is now no longer left alone and uninjured, it has superposed upon itself a more or less strong reflected wave, a reflected wave which constantly increases in intensity as the distance along the cable, or the penetration of the wave into a conducting medium, increases; all the elementary reflected waves get mixed up by re-reflexion in the rear, constituting what Mr. Heaviside calls a diffusive "tail"; and this accumulation of reflected waves it is which constitutes what is known as "distortion" in cables, and what is known as "opacity" inside conducting dielectrics.

There is another kind of opacity, a kind due to heterogeneousness, not connected with conductivity but due merely to a change in the constants K and μ,—properly a kind of translucency, a scattering but not a dissipation of energy,—like the opacity of foam or ground glass.

This kind of opacity is an affair of boundaries and not of the medium itself, but after all, as we now see, it has features by no means altogether dissimilar to the truer kind of opacity. Conducting opacity is due to reflexion, translucent opacity is due to reflexion,—to irregular reflexion as it is called, but of course there is nothing irregular about the reflexion, it is only the distribution of boundaries which is complicated, the reflexion is as simple as ever;—except, indeed, to some extent when the *size* of the scattering particles has to be taken into account and the blue of the sky emerges. But my point is that this kind of opacity also is after all of the reflexion kind, and the gradual destruction of the advancing wave—whether it be by dust in the air or, as Lord Rayleigh now suggests, perhaps by the discrete molecules themselves, by the same molecular property as causes refraction and dispersion—must result in a minute distortion and a mode of wave propagation not wholly different from cable-signalling or from the transmission of light through conductors. So that the red of the sunset sky and the green of gold-leaf may not be after all very different; nor is the arrival-curve of a telegraph signal a wholly distinct phenomenon.

There is a third kind of opacity, that of lampblack, where the molecules appear to take up the energy direct, converting

it into their own motion, that is into heat, and where there appears to be little or nothing of the nature of reflexion. I am not prepared to discuss that kind at present.

It is interesting to note that in the most resisting and capacious cable that ever was made, where all the features of every wave arrive as obliterated as if one were trying to signal by heat-pulses through a slab, that even there the *head* of every wave travels undistorted, with the velocity of light, and suffers nothing but attenuation; for the superposed reversed field is only called out by the arrival of the direct pulse, and never absolutely reaches the strength of the direct field. The attenuation may be excessive, but the signal is there in its right time if only we have a sensitive enough instrument to detect it; though it would be practically useless as a signal in so extreme a case, being practically *all* tail.

Nothing at all reaches the distant end till the light-speed-time has elapsed; and the light-speed-time in a cable depends on the μ and K of its insulating sheath, depends, if that is not simply cylindrical, on the product of its self-inductance and capacity per unit length; but at the expiration of the light-speed-time the head of the signalling pulse arrives, and neither wire-resistance nor insulation-leakage, no, nor magnetic-conductivity, can do anything either to retard it or to injure its sharpness: they can only enfeeble its strength, but they can do that very effectually.

The transmitter of the pulse is self-induction in conjunction with capacity: the chief practical enfeebler of the pulse is wire-resistance in conjunction with capacity; and before Atlantic telephony is possible (unless a really distortionless cable is forthcoming) the copper core of an ordinary cable will have to be made much larger. Nothing more is wanted in order that telephony to America may be achieved. There may be practical difficulties connected with the mechanical stiffness of a stout core and the worrying of its guttapercha sheath, and these difficulties may have to be lessened by aiming at distortionless conditions—it is well known also that for high frequencies a stout core must be composed of insulated strands unless it is hollow—but when such telephony is accomplished, I hope it will be recollected that the full and complete principles of it and of a great deal else connected with tele-

graphy have been elaborately and thoroughly laid down by Mr. Heaviside.

There is a paragraph in Maxwell, concerning the way a current rises in a conductor and affects the surrounding space, which is by no means satisfactory: it is Art. 804. He takes the current as starting all along the wire, setting up a sheath of opposition induced currents in the surrounding imperfectly insulating dielectric, which gradually diffuse outwards and die away, leaving at last the full inductive effect of the core-current to be felt at a distance. Thus there is supposed to be a diffusion of energy outwards from the wire, which he likens to the diffusion of heat.

But, as Mr. Heaviside has shown, the true phenomenon is the transmission of a wave in the space surrounding the wire—a plane wave if the wire is perfectly conducting, a slightly coned wave if it resists,—a wave-front perpendicular to the wire and travelling along it,—*a sort of beam of dark light with the wire as its core.*

Telegraphic signalling and optical signalling are similar; but whereas the beam of the heliograph is abandoned to space and must go straight except for reflexion and refraction, the telegraphic beam can follow the sinuosities of the wire and be guided to its destination.

If the medium conducts slightly it will be dissipated *in situ*; but if the wire conducts imperfectly, a minute trickle of energy is constantly directed inwards radially towards the wire core, there to be dissipated as heat. Parallel to the wire flows the main energy stream, but there is a small amount of tangential grazing and inward flow. The initial phenomenon does not occur in the wire, gradually to spread outwards, but it occurs in the surrounding medium, and a fraction of it gradually converges inwards. The advancing waves are not cylindrical but plane waves, and though the diffusing waves are cylindrical they advance inwards, not outwards.

I will quote from a letter of Mr. Heaviside's:—"The easiest way to make people understand is, perhaps, to start with a conducting dielectric with plane waves in it without wires, [thus getting] one kind of attenuation and distortion. Then introduce wires of *no* resistance; there is no difference except

in the way the lines of force distribute [enabling the wires to guide the plane waves]. Then introduce magnetic conductivity in the medium, [thereby getting] the other kind of attenuation and distortion. Transfer it to the wires, making it electrical resistance. Then abolish the first electric conductivity, and you have the usual electric telegraph."

Opacity of Gold-leaf.

Now returning to the general solution (5) let us apply it to calculate the opacity of gold-leaf to light.

Take $\sigma = 2000\,\mu$ square centim. per sec.,
$p = 2\pi \times 5 \times 10^{14}$ per sec.;

then the critical quantity $4\pi/p\sigma K$ or (7) is

$$\frac{2 \times 9 \times 10^{20}}{5 \times 10^{14} \times 2000\, K/K_0} = \frac{1800}{K/K_0}.$$

This number is probably considerably bigger than unity (unless, indeed, the specific inductive capacity K/K_0 of gold is immensely large, which may indeed be the case—refractive index 40, for instance,—only it becomes rather difficult to define); so that, approximately,

$$\alpha = \sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)} = \sqrt{\frac{40 \times 5 \times 10^{14}}{2000}} = \sqrt{10^{13}} = 3 \times 10^{6};$$

or the damping distance is

$$\tfrac{1}{30} \times 10^{-5} \text{ centim.} = \tfrac{1}{3} \text{ microcentimetre},$$

whereas the wave-length in air is

$$6 \times 10^{-5} \text{ centim.} = 60 \text{ microcentimetres.}$$

The damping distance is therefore getting nearer to the right order of magnitude, but the opacity is still excessive.

A common thickness for gold-leaf is stated to be half a wave-length of light; that is to say, 90 times the damping distance. Hence the amplitude of the light which gets through a half-wave thickness of gold is e^{-90} of that which enters; and that is sheer opacity.

[Maxwell's calculation in Art. 798, carried out numerically, makes the damping

$$e^{-\frac{2\pi\mu o}{\sigma}x}, \;=\; \exp.\,(-10^8 x) \text{ for gold,}$$

see equation (8) above; or, for a thickness of half a wave-length, 10^{-1000}, which is billions of billions of billions (indeed a number with 960 digits) times greater opacity than what we have here calculated, and is certainly wrong.]

It must, however, be granted, I think, that the green light that emerges from gold-leaf is not properly transmitted; it is light re-emitted by the gold *. The incident light, say the red, is all stopped by a thickness less than half a wave-length. The green light may conceivably be due to atoms vibrating fairly in concordance, and not calling out the conducting opacity of the metal. If the calculated opacity, notwithstanding this, is still too great, it is no use assuming a higher conductivity at higher frequency, for that would act the wrong way. What must be assumed is either some special molecular dispersion theory, or else greater specific resistance for oscillations of the frequency which get through; nor must the imaginative suggestion made immediately below equation (13) be altogether lost sight of.

There is, however, the possibility mentioned above that the relative specific inductive capacity of gold, $K K_0$, if a meaning can be attached to it, may be very large, perhaps (though very improbably, see Drude, Wied. *Ann.* vol. xxxix. p. 481) comparable with 1800. Suppose for a moment that it is equal to 1800; then the value of the critical quantity (7) is 1 and the value of α is

$$p\sqrt{\tfrac{1}{2}\mu K \times \cdot 4} = \frac{p}{v}\sqrt{\left(\frac{K}{5K_0}\right)} = \frac{2\pi \times 5 \times 10^{14}}{3 \times 10^{10}}\sqrt{360}$$
$$= 19 \times 10^5,$$

which reduces the calculated opacity considerably, though still not enough.

In general, calling $K/K_0 = c$, and writing the critical number $\frac{4\pi\mu v^2}{\sigma p c}$ as h/c, we have

$$\frac{2v^2}{p^2}\alpha^2 = \tfrac{1}{2}\alpha^2/\beta^2 = \lambda^2\alpha^2/2\pi^2 = \sqrt{(h^2+c^2)} - c\,;$$

* This would be fluorescence, of course; and Dr. Larmor argues in favour of a simple ordinary exponential coefficient of absorption even in metals. See Phil. Trans. 1894, p. 738, § 27.

so that $a\lambda/2\pi$ ranges from $\sqrt{\frac{1}{2}h}$ when h/c is big, to $\frac{1}{2}h/\sqrt{c}$ when h/c is small.

Writing the critical number h/c as $\tan\epsilon$, the general value of a is given by

$$a\lambda = \pi\sqrt{2c(\sec\epsilon - 1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This is the ratio of the wave-length in air to the damping distance in the material in general; meaning by "the damping distance" the thickness which reduces the amplitude in the ratio $e:1$. (14) represents expression (5); compare with (12′).

Theory of a Film.

So far nothing has been said about the limitation of the medium in space, or the effect of a boundary, but quite recently Mr. Heaviside has called my attention to a special theory, a sort of Fresnel-like theory, which he has given for infinitely thin films of finite conductance; it is of remarkable simplicity, and may give results more in accordance with experiment than the theory of the universal opaque medium without boundary, hitherto treated: a medium in which really the source is immersed.

Let a film, not so thick as gold-leaf, but as thin as the black spot of a soap-bubble, be interposed perpendicularly between source and receiver. I will quote from 'Electrical Papers,' vol. ii. p. 385:—"Let a plane wave $E_1 = \mu v H_1$ moving in a nonconducting dielectric strike flush an exceedingly thin sheet of metal [so thin as to escape the need for attending to internal reflexions, or the double boundary, or the behaviour inside]; let $E_2 = \mu v H_2$ be the transmitted wave out in the dielectric on the other side, and $E_3 = -\mu v H_3$ be the reflected wave *.

* *General Principles.*—It may be convenient to explain here the principles on which Mr. Heaviside arrives at his remarkably neat expression for a wave-front in an insulating medium,

$$E = \mu v H,$$

or as it may be more fully and vectorially written,

$$V(\nu \mathbf{E}) = \mu v \mathbf{H},$$

where $\mathbf{E}$ is a vector representing the electric intensity (proportional to the electric displacement), $\mathbf{H}$ is the magnetic intensity, and ν is unit normal to the wave-front. $\mathbf{E}$ and $\mathbf{H}$ are perpendicular vectors in the

"At the sheet we have

$$E_1 + E_3 = E_2$$
$$H_1 + H_3 = H_2 + 4\pi k z E_2,$$

k being the conductivity of the sheet of thickness z. Therefore

$$\frac{E_2}{E_1} = \frac{H_2}{H_1} = \frac{E_1 + E_3}{E_1} = \frac{1}{1 + 2\pi\mu k z v}, \quad . \quad . \quad (15)$$

same plane, *i.e.* in the same phase, and **E H** ν are all at right angles to each other.

The general electromagnetic equations in an insulating medium are perhaps sufficiently well known to be, on Mr. Heaviside's system,

$$\text{curl}\,\mathbf{H} = \text{K}\dot{\text{E}} \quad \text{and} \quad -\text{curl}\,\mathbf{E} = \mu\dot{\mathbf{H}},$$

where "curl" is the vector part of the operator ∇, and where Maxwell's vector-potential and other complexities have been dispensed with.

[In case these equations are not familiar to students I interpolate a parenthetical explanation which may be utilised or skipped at pleasure.

The orthodox definition of Maxwell's name "curl" is that **b** is called the curl of **a** when the surface-integral of **b** through an area is equal to the line-integral of **a** round its boundary, **a** being a vector or a component of a vector agreeing everywhere with the boundary in direction, and **b** being a vector or component of vector everywhere normal to the area. Thus it is an operator appropriate to a pair of looped or interlocked circuits, such as the electric and the magnetic circuits always are. The first of the above fundamental equations represents the fact of electromagnetism, specially as caused by displacement currents in an insulator, the second represents the fact of magneto-electricity, Faraday's magneto-electric induction, in any medium. Taking the second first, it states the fundamental law that the induced EMF in a boundary equals the rate of change in the lines of force passing through it; since the EMF or step of potential all round a contour is the line-integral of the electric intensity E round it, so that

$$\text{EMF} = \int_{\text{cycle}} \text{E}ds = -\frac{d\text{N}}{dt} = -\iint \dot{\text{B}}d\text{S} = -\iint \mu\dot{\text{H}}d\text{S}\,;$$

wherefore $-\mu\dot{\text{H}}$ equals the curl of E. (The statement of this second circuital law is entirely due to Mr. Heaviside; it is now largely adopted and greatly simplifies Maxwell's treatment, abolishing the need for vector potential.) [See, however, Appendix III. p. 386. *May* 1899.]

The first of the above two fundamental equations, on the other hand, depends on the fact that a current round a contour excites lines of magnetic force through the area bounded by it, and states the law that the total magnetomotive force, or line-integral of the magnetic intensity round the boundary, is equal to 4π times the total current through it; the total current being the "ampere-turns" of the practical Engineer.

H is reflected positively and E negatively. A perfectly conducting barrier is a perfect reflector; it doubles the magnetic force and destroys the electric force on the side containing the incident wave, and transmits nothing."

[I must here interpolate a remark to the effect that though it can hardly be doubted that the above boundary conditions (tangential continuity of both E and H) are correct, yet in general we cannot avoid some form of æther-theory

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Expressing this law in terms of current density c, we write

$$\text{MMF} = \int_{\text{cycle}} \text{H}ds = 4\pi\text{C} = \iint 4\pi c d\text{S};$$

so always current-density represents the curl of the magnetic field due to it, or curl $\text{H}=4\pi c$.

Now in a conductor $c=k\text{E}$, but in an insulator $c=\dot{\text{D}}$, the rate of change of displacement or Maxwell's "displacement-current"; and the displacement itself is proportional to the intensity of the electric field, $\text{D}=\frac{\text{K}}{4\pi}\text{E}$; hence the value of current density in general is

$$c=k\text{E}+\frac{\text{K}}{4\pi}\dot{\text{E}},$$

whence in general

$$\text{curl H}=4\pi k\text{E}+\text{K}\dot{\text{E}}=(4\pi k+\text{K}p)\text{E},$$

and in an insulator the conductivity k is nothing.

The connexion between "curl" so defined and $\text{V}\nabla$ is explained as follows. The operator ∇ applied to a vector R whose components are X Y Z gives

$$\left(i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}\right)(i\text{X}+j\text{Y}+k\text{Z}),$$

which, worked out, yields two parts

$$\text{S}\nabla\text{R} \text{ or } -\left(\frac{d\text{X}}{dx}+\frac{d\text{Y}}{dy}+\frac{d\text{Z}}{dz}\right),$$

also called convergence, and

$$\text{V}\nabla\text{R} \text{ or } i\left(\frac{d\text{Z}}{dy}-\frac{d\text{Y}}{dz}\right)+j\left(\frac{d\text{X}}{dz}-\frac{d\text{Z}}{dx}\right)+k\left(\frac{a\text{Y}}{dx}-\frac{d\text{X}}{dy}\right)$$

or say $i\xi+j\eta+k\zeta$, where $\xi\ \eta\ \zeta$ are the components of a spin-like vector ω. Now a theorem of Sir George Stokes shows that the normal component of ω integrated over any area is equal to the tangential component of R integrated all round its boundary; hence $\text{V}\nabla$ and curl are the same thing.

Whenever ω is zero it follows that R has no circulation but is the derivative of an ordinary single-valued potential function, whose $d\text{V}=\text{X}dx+\text{Y}dy+\text{Z}dz$. In electromagnetism this condition is by no means

when we have to lay down continuity conditions, and, according to the particular kind of æther-theory adopted so will the boundary conditions differ. My present object is to awaken a more general interest in the subject and to represent Mr. Heaviside's treatment of a simple case; but it must be understood that the continuity conditions appropriate to oblique incidence have been treated by other great mathematical physicists, notably by Drude, J. J. Thomson, and

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satisfied. $\dot{E}$ and H or $\dot{H}$ and E are both full of circulation, and their circuits are interlaced. Fluctuation in E by giving rise to current causes H; fluctuation in H causes induced E.]

Now differentiating only in a direction normal to a plane wave advancing along x, the operator $V\nabla$ becomes simply id/dx when applied to any vector *in* the wave-front, the scalar part of ∇ being nothing.

So the second of the above fundamental equations can be written

$$-i\frac{dE}{dx}=\mu\frac{dH}{dt},$$

or

$$\frac{dE}{dH}=\mu iv;$$

so, ignoring any superposed constant fields of no radiation interest, E and H are vectors in the same phase at right angles to each other, and their tensors are given by $E=\mu v H$.

Similarly of course the other equation furnishes $H=KvE$; thus giving the ordinary $K\mu v^2=1$, and likewise the fact that the electric and magnetic energies per unit volume are equal, $\frac{1}{2}KE^2=\frac{1}{2}\mu H^2$.

A wave travelling in the opposite direction will be indicated by $E=-\mu vH$; hence, as is well known, if either the electric or the magnetic disturbance is reversed in sign the direction of advance is reversed too.

(The readiest way to justify the equation $E=\mu vH$, *à posteriori*, is to *assume* the two well-known facts obtained above, viz. that the electric and magnetic energies are equal in a true advancing wave, and that $v=1/\sqrt{\mu K}$; then it follows at once.)

Treatment of an insulating boundary.—At the boundary of a different medium without conductivity the tangential continuity of E and of H across the boundary gives us the equations

$$E_1+E_3=E_2$$
$$H_1+H_3=H_2,$$

where the suffix 1 refers to incident, the suffix 2 to transmitted, and the suffix 3 to reflected waves.

H_1+H_3 may be replaced by $\mu v(E_1-E_3)$, since the reflected wave is

Larmor, also by Lord Rayleigh, and it would greatly enlarge the scope of this Address if I were to try to discusss the difficult and sometimes controversial questions which arise. I must be content to refer readers interested to the writings of the Physicists quoted—especially I may refer to J. J. Thomson's 'Recent Researches,' Arts. 352 to 409, and to Larmor, Phil. Trans, 1895, vol. 186, Art. 30, and other places.]

Now apply this to an example. Take k for gold, as we have done before, to be $1/2000\,\mu$ seconds per square centim.

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reversed; so we shall have, for the second of the continuity equations,

$$E_1 - E_3 = \frac{\mu_2 v_2}{\mu_1 v_1} E_2 = nmE_2 ;$$

n being the index of refraction, and m the relative inductivity. Hence, adding and subtracting,

$$\frac{E_2}{E_1} = \frac{2}{1+nm},$$

and

$$\frac{E_3}{E_1} = -\frac{1-nm}{1+nm};$$

well-known optical expressions for the transmitted and reflected amplitudes at perpendicular incidence, except that the possible magnetic property of a transparent medium is usually overlooked.

Treatment of a conducting boundary.—But now, if the medium on the other side of the boundary is a conductor instead of a dielectric, a term in one of the general equations must be modified; and, instead of curl $H = KpE$, we shall have, as the fundamental equation inside the medium,

$$-\frac{dH}{dx} = 4\pi kE;$$

or more generally $(4\pi k + Kp)E$.

So, on the far side of a thin slice of thickness z, the magnetic intensity H_2 is not equal to the intensity $H_1 + H_3$ on the near side, but is less by

$$dH = 4\pi kEdx = 4\pi kE_2 z = 4\pi k\mu vzH_2 ;$$

and this explains the second of the continuity equations immediately following in the text.

In a quite general case, where all the possibilities of conductivity and capacity &c. are introduced at once, the ratio of E/H is not μv or $(\mu/K)^{\frac{1}{2}}$, but is $(g+\mu p)^{\frac{1}{2}}(4\pi k+Kp)^{-\frac{1}{2}}$ for waves in a general material medium, (g may always be put zero), or $(R+pL)^{\frac{1}{2}}(Q+pS)^{-\frac{1}{2}}$ for waves guided by a resisting wire through a leaky dielectric.

The addition of dielectric capacity to conductivity in a film is therefore simple enough and results in an equation quoted in the text below.

and $v=3\times10^{10}$ centim. per sec., for v is the velocity in the dielectric not in the conductor; then take a film whose thickness z is one twenty-fifth of a wave-length of the incident light; and the ratio of the transmitted to the incident amplitude comes out

$$1/2\pi\mu kvz=\frac{1000}{\pi vz}=\frac{1}{200}.$$

Some measurements made by W. Wien at Berlin in 1888 (Wied. *Ann.* vol. xxxv.), with a bunsen-burner as source of radiation, give as the actual proportion of the transmitted to the long-wave incident light, for gold whose thickness is

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Simple treatment of the E.M. theory of light.—It is tempting to show how rapidly the two fundamental electromagnetic equations, in Mr. Heaviside's form, lead to the electromagnetic theory of light, if we attend specially to the direction normal to the plane of the two perpendicular vectors E and H, to the direction along say x, so that $\nabla = i\,d/dx$ and $\nabla^2 = -d^2/dx^2$.

In an insulating medium the equations are

$$\text{curl}\,\mathrm{H}=\mathrm{K}\dot{\mathrm{E}} \qquad \text{and} \qquad -\text{curl}\,\mathrm{E}=\mu\dot{\mathrm{H}};$$

now curl$=\mathrm{V}\nabla=\nabla$, since $\mathrm{S}\nabla=0$ in this case, so

$$\nabla^2\mathrm{H}=\mathrm{K}\nabla\dot{\mathrm{E}}=\mathrm{K}\,\text{curl}\,\dot{\mathrm{E}}=-\mathrm{K}\mu\ddot{\mathrm{H}};$$

or, in ordinary form,

$$\frac{d^2\mathrm{H}}{dx^2}=\mathrm{K}\mu\frac{d^2\mathrm{H}}{dt^2},$$

and there are the waves.

If this is not rigorous, there is no difficulty in finding it done properly in other places. I believe it to be desirable to realize things simply as well.

In a conducting medium the fundamental equations are, one of them,

$$\text{curl}\,\mathrm{H}=\mathrm{K}\dot{\mathrm{E}}+4\pi k\mathrm{E}=(\mathrm{K}p+4\pi k)\mathrm{E},$$

while the other remains unchanged; unless we like to introduce the non-existent auxiliary g, which would make it

$$-\mathrm{V}\nabla\mathrm{E}=(g+\mu p)\mathrm{H},$$

and would cover wires too.

So
$$-\nabla^2\mathrm{H}=(4\pi k+\mathrm{K}p)(g+\mu p)\mathrm{H},$$

the general wave equation. In all these equations p stands for d/dt; but, for the special case of simply harmonic disturbance of frequency $p/2\pi$, of course ip can be substituted.

10^{-5} centim., ·0033 or 1/300; while for gold one quarter as thick the proportion was 0·4 (see Appendix II. page 385).

He tried also two intermediate thicknesses, and though approximately the opacity increases with the square of the thickness, it really seems to increase more rapidly: as no doubt it ought, as the boundaries separate. However, for a thickness $\lambda/25$ I suppose we may assume that about 1/3rd of the light would be transmitted, whereas the film-theory gives $(1/200)^2$; so even now a metal calculates out too opaque though it is rather less hopelessly discrepant than it used to be. The result, we see, for the infinitely thin film, is independent of the frequency.

Specific inductive capacity has not been taken into account in the metal, but if it is it does not improve matters. It does not make much difference, unless very large, but what difference it does make is in the direction of increasing opacity. In a letter to me Mr. Heaviside gives for the opacity of a film of highly conducting dielectric

$$\frac{E_1}{E_2}=\left\{(1+2\pi\mu kvz)^2+(\tfrac{1}{2}mczp/v)^2\right\}, \quad . \quad . \quad (16)$$

where I have replaced his $\frac{1}{2}\mu vz\mathrm{K}p$ last term by an expression with the merely relative numbers K/K_0 and μ/μ_0, called c and m respectively, thus making it easier to realise the magnitude of the term, or to calculate it numerically.

Theory of a Slab.

An ordinary piece of gold-leaf, however, cannot properly be treated as an infinitely thin film; it must be treated as a slab, and reflexions at its boundaries must be attended to. Take a slab between $x=0$ and $x=l$. The equations to be satisfied inside it are the simplified forms of the general fundamental ones

$$-\frac{d\mathrm{E}}{dx}=\mu p\mathrm{H}, \qquad -\frac{d\mathrm{H}}{dx}=4\pi k\mathrm{E},$$

k being $1/\sigma$, and K being ignored; while outside, at $x=l$, the condition $\mathrm{E}=\mu v\mathrm{H}$ has to be satisfied, in order that a wave may emerge.

The following solutions do all this if $q^2 = 4\pi\mu kp$:—

$$E = Ae^{qx}\left(1 + \frac{qv+p}{qv-p}e^{2q(l-x)}\right),$$

$$\mu v H = -\frac{Aqv}{p}e^{qx}\left(1 - \frac{qv+p}{qv-p}e^{2q(l-x)}\right).$$

Conditions for the continuity of both E and H at $x=0$ suffice to determine A, namely if E_1 H_1 is the incident and E_3 H_3 the reflected wave on the entering side, while E_0 H_0 are the values just inside, obtained by putting $x=0$ in the above,

$$E_1 + E_3 = E_0,$$
$$E_1 - E_3 = \mu v(H_1 + H_3) = \mu v H_0.$$

Adding, we get a value for A in terms of the incident light E_1,

$$2E_1 p(qv-p) = A\{(qv+p)^2 e^{2ql} - (qv-p)^2\}.$$

whence we can write E anywhere in the slab,

$$\frac{E}{E_1} = \frac{2p(qv-p)}{(qv+p)^2 e^{2ql} - (qv-p)^2}\left\{e^{qx} + \frac{qv+p}{qv-p}e^{2ql}e^{-qx}\right\}.$$

Put $x=l$, and call the emergent light E_2; then

$$\frac{E_2}{E_1} = \frac{4pqv}{(qv+p)^2 e^{ql} - (qv-p)^2 e^{-ql}} = \rho, \text{ say}, \quad . \quad (17)$$

and this constitutes the measure of the opacity of a slab, ρ^2 being the proportion of incident light transmitted.

It is not a simple expression, because of course p signifies the operator d/dt, and though it becomes simply ip for a simply harmonic disturbance, yet that leaves q complex. However, Mr. Heaviside has worked out a complete expression for ρ^2, which is too long to quote (he will no doubt be publishing the whole thing himself before long), but for slabs of considerable opacity, in which therefore *multiple* reflexions may be neglected, the only important term is

$$\rho = \frac{4\sqrt{2}\, e^{-\alpha l} p/\alpha v}{1 + (1 + p/\alpha v)^2}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

with

$$\alpha = \sqrt{(2\pi\mu p/\sigma)} = 3 \times 10^6$$

for light in gold; and

$$\frac{p}{\alpha v}=\frac{2\pi}{\alpha\lambda}=\frac{2\pi}{30\times 5}=\frac{1}{25}\text{ about.}$$

So the effect of attending to reflexion at the walls of the slab is to still further diminish the amplitude that gets through, below the $e^{-\alpha l}$ appropriate to the unbounded medium, in the ratio of $\frac{2\sqrt{2}}{25}$, or about a ninth.

Effect of each Boundary.

It is interesting to apply Mr. Heaviside's theory to a study of what happens at the first boundary alone, independent of subsequent damping.

Inside the metal, by the two fundamental equations, we have

$$E_0=\left(\frac{\mu p}{4\pi k}\right)^{\frac{1}{2}}H_0,$$

and by continuity across the boundary

$$E_1+E_3=E_0,$$
$$E_1-E_3=\mu vH_0 \quad =\mu vE_0\left(\frac{4\pi k}{\mu p}\right)^{\frac{1}{2}}=\frac{qv}{p}E_0,$$

where still $q^2=4\pi\mu kp$.

Therefore, for the transmitted amplitude

$$\frac{E_0}{E_1}=\frac{2p}{p+qv},$$

and for the reflected

$$\frac{E_3}{E_1}=\frac{p-qv}{p+qv},$$

or rationalising and writing amplitudes only, and understanding by p no longer d/dt in general, but only 2π times the frequency,

$$\frac{E_0}{E_1}=\frac{2p}{\sqrt{((p+\alpha v)^2+\alpha^2v^2)}}=\rho_1\text{ say.} \quad . \quad (19)$$

Any thickness of metal multiplies this by the factor $e^{-\alpha x}$, and then comes the second boundary, which, according to what has been done above, has a comparatively small but peculiar effect; for it ought to change the amplitude from ρ_1

into ρ, that is to give an emergent amplitude

$$\frac{4\sqrt{2}\,.\,p/\alpha v}{1+(1+p/\alpha v)^2}\,E_1 e^{-\alpha l},$$

instead of the above incident on the second boundary

$$\frac{2p/\alpha v}{\sqrt{(1+(1+p/\alpha v)^2)}}\,E_1 e^{-\alpha l}\,; \quad . \quad . \quad (20)$$

that is for the case of light in gold, for which $p/\alpha v$ is small, to change $2/\sqrt{2}$ into $2\sqrt{2}$, in other words, to double it.

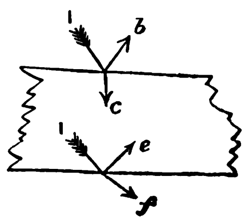
The effect of the first boundary alone, ρ_1, is $\frac{2\sqrt{2\pi}}{\alpha\lambda}$, or say 1/18, and this is a greater reduction effect than that reckoned above for the two boundaries together.

Thus the obstructive effect of the two boundaries together comes out less than that of the first boundary alone—an apparently paradoxical result. About one-eighteenth of the light-amplitude gets through the first boundary, but about one-ninth gets through the whole slab (ignoring the geometrically progressive decrease due to the thickness, that is ignoring $e^{-\alpha l}$, and attending to the effect of the boundaries alone; which, however, cannot physically be done). At first sight this was a preposterous and ludicrous result. The second or outgoing boundary ejects from the medium nearly double the amplitude falling upon it from inside the conductor! But on writing this, in substance, to Mr. Heaviside he sent all the needful answer by next post. "The incident disturbance inside is not the whole disturbance inside."

That explains the whole paradox—there is the reflected beam to be considered too. At the *entering* boundary the incident and reflected amplitudes are in opposite phase, and nearly equal, and their algebraic sum, which is transmitted, is small. At the *emerging* boundary the incident and reflected amplitudes are in the same phase, and nearly equal, and their algebraic sum, which is transmitted, is large—is nearly double either of them. But it is a curious action:—either more light is pushed out from the limiting boundary of a conductor than reaches it inside, or else, I suppose, the velocity of light inside the metal must be greater than it is outside, a result not contradicted by Kundt's refraction experiments, and suggested by most optical

theories. It is worth writing out the slab theory a little more fully, to make sure there is no mistake, though the whole truth of the behaviour of bodies to light can hardly be reached without a comprehensive molecular dispersion theory. I do not think Mr. Heaviside has published his slab theory anywhere yet. A slab theory is worked out by Prof. J. J. Thomson in Proc. Roy. Soc. vol. xlv., but it has partly for its object the discrimination between Maxwell's and other rival theories, so it is not very simple. Lord Kelvin's Baltimore lectures probably contain a treatment of the matter. All that I am doing, or think it necessary to do in an Address, is to put in palatable form matter already to a few leaders likely to be more or less known: in some cases perhaps both known and objected to.

The optical fractions of Sir George Stokes, commonly written $b\ c\ e\ f$, are defined, as everyone knows, as follows. A ray falling upon a denser body with incident amplitude 1 yields a reflected amplitude b and a transmitted c. A ray falling upon the boundary of a rare body with incident amplitude 1 has an internally reflected amplitude e and an emergent f. General principles of reversibility show that $b+e=0$, and that $b^2+cf=1$ in a transparent medium.



Now in our present case we are attending to perpendicular incidence only, and we are treating of a conducting slab; indeed, we propose to consider the obstructive power of the material of the slab so great that we need not suppose that any appreciable fraction of light reflected at the second surface returns to complicate matters at the first surface. This limitation by no means holds in Mr. Heaviside's complete theory, of course, but I am taking a simple case.

The characteristic number which governs the phenomenon is $\frac{p}{av}$ or $\frac{2\pi}{a\lambda}$, a number which for light and gold we reckoned as being about $\frac{1}{27}$, that is decidedly smaller than unity, a being $\sqrt{(2\pi\mu kp)}$ or $\sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)}$. The characteristic number p/av

we will for brevity write as h, and we will express amplitudes for perpendicular incidence only, as follows:—

Incident amplitude 1,

$$\text{externally reflected} \quad b = -\left\{\frac{1+(1-h)^2}{1+(1+h)^2}\right\}^{\frac{1}{2}}$$

$$\text{entering} \quad c = \frac{2h}{\{1+(1+h)^2\}^{\frac{1}{2}}},$$

Incident again 1,

$$\text{internally reflected} \quad e = \left\{\frac{1+(1-h)^2}{1+(1+h)^2}\right\}^{\frac{1}{2}},$$

$$\text{emergent} \quad f = \frac{2\sqrt{2}}{\{1+(1+h)^2\}^{\frac{1}{2}}}.$$

(It must be remembered that e and f refer to the second boundary *alone*, in accordance with the above diagram.)

Thus the amplitude transmitted by the whole slab, or rather by both surfaces together, ignoring the opacity of its material for a moment, is

$$\text{transmitted} \quad cf = \frac{4\sqrt{2}\,.\,h}{1+(1+h)^2}.$$

To replace in this the effect of the opaque material, of thickness l, we have only to multiply by the appropriate exponential damper, so that the amplitude ultimately transmitted by the slab is

$$\frac{4\sqrt{2}\,.\,p/\alpha v}{1+(1+p/\alpha v)^2}e^{-\alpha l}$$

times the amplitude originally incident on its front face.

This agrees with the expression (18) specifically obtained above for this case, but, once more I repeat, multiple reflexions have for simplicity been here ignored, and the medium has been taken as highly conducting or very opaque.

But even so the result is interesting, especially the result for f. To emphasize matters, we may take the extreme case when the medium is so opaque that h is nearly zero; then b is nearly -1, c is nearly 0, being $h\sqrt{2}$, e is the same as b except for sign, and f is nearly 2.

An opaque slab transmits $8h^2e^{-2\alpha l}$ of the incident light energy; its first boundary transmits only $2h^2$. The second

or emergent boundary doubles the amplitude. Taken in connexion with the facts of selective absorption and the timing of molecules to vibrations of certain frequency, I think that this fact can hardly be without influence on the green transparency of gold-leaf.

APPENDIX I.

MR. HEAVISIDE'S *Note on Electrical Waves in Sea-Water.*

[Contributed to a discussion at the Physical Society in June 1897: see Mr. Whitehead's paper, Phil. Mag. August 1897.]

"To find the attenuation suffered by electrical waves through the conductance of sea-water, the first thing is to ascertain whether, at the frequency proposed, the conductance is paramount, or the permittance, or whether both must be counted.

"It is not necessary to investigate the problem for any particular form of circuit from which the waves proceed. The attenuating factor for plane waves, due to Maxwell, is sufficient. If its validity be questioned for circuits in general, then it is enough to take the case of a simply-periodic point source in a conducting dielectric ('Electrical Papers,' vol. ii. p. 422, § 29). The attenuating constant is the same, viz. (equation (199) *loc cit.*):—

$$n_1 = \frac{n}{v} \sqrt{\tfrac{1}{2}} \left[\left\{ 1 + \left(\frac{4\pi k}{cn} \right)^2 \right\}^{\frac{1}{2}} - 1 \right]^{\frac{1}{2}},$$

where $n/2\pi$ is the frequency, k the conductivity, c the permittivity, and $v = (\mu c)^{-\frac{1}{2}}$, μ being the inductivity.

"The attenuator is then $e^{-n_1 r}$ at distance r from the source, as in plane waves, disregarding variations due to natural spreading. It is thus proved for any circuit of moderate size compared with the wave-length, from which simply periodic waves spread.

"The formula must be used in general, with the best values of k and c procurable. But with long waves it is pretty certain that the conductance is sufficient to make $4\pi k/cn$ large. Say with common-salt-solution $k = (30^{11})^{-1}$, then

$$\frac{4\pi k}{cn} = \frac{2k\mu v^2}{f}$$

if f is the frequency. This is large unless f is large, whether we

assume the specific c/c_0 to have the very large value 80 or the smaller value effectively concerned with light waves. We then reduce n_1 to

$$n_1=(2n\mu k\pi)^{\frac{1}{2}}=2\pi(\mu kf)^{\frac{1}{2}},$$

as in a pure conductor.

"This is practically true perhaps even with Hertzian waves, of which the attenuation has been measured in common-salt-solution by P. Zeeman. If then $k^{-1}=30^{11}$ [and if the frequency is 300 per second] we get n_1=about $\frac{1}{5000}$.

"Therefore 50 metres is the distance in which the attenuation due to conductivity is in the ratio 2·718 to 1, and there is no reason why the conductivity of sea-water should interfere, if the value is like that assumed above.

"These formulæ and results were communicated by me to Prof. Ayrton at the beginning of last year, he having enquired regarding the matter, on behalf of Mr. Evershed I believe.

"The doubtful point was the conductivity. I had no data, but took the above k from a paper which had just reached me from Mr. Zeeman. Now Mr. Whitehead uses $k^{-1}=20^{10}$, which is no less than 15 times as great. I presume there is good authority for this datum *. None is given. Using it we obtain $n_1=\frac{1}{1316}$.

"Thus 50 metres is reduced to 13·16 metres. But a considerably greater conductivity is required before it can be accepted that the statements which have appeared in the press, that the failure of the experiments endeavouring to establish telegraphic communication with a light-ship from the sea-bottom was due to the conductance of the sea, are correct. It seems unlikely theoretically, and Mr. Stevenson has contradicted it (in 'Nature') from the practical point of view. So far as I know, no account has been published of these experiments, therefore there is no means of finding the cause of the failure."

* Dr. J. L. Howard has recently set a student to determine the resistivity of the sea-water used by Professor Herdman, density 1·019 gr. per c.c., and he finds it to be 3×10^{10} c.g.s. at 15° C.—O. J. L., March 1899.

Added, May 1899.—An old student of mine, Prof. W. M. Thornton, of the Durham College, Newcastle, tells me that the resistance of sea-water collected at Tynemouth pier at flood-tide is $2{\cdot}39\times10^{10}$ c.g.s. at 15° C.

Appendix II.

Experiments on the Transparency of Metals.

The experiments of W. Wien on the transparency of metals, by means of a bolometer arranged to receive the radiation from a bunsen burner transmitted through different films, resulted in the following numbers for the proportion of radiation transmitted.

Metal.	Thickness in 10^{-7} centim.	Proportion transmitted.		Proportion reflected.
		Bunsen burner luminous.	Bunsen burner non-luminous.	
Platinum	20	·32	·37	·13
Iron & Platinum	40+20	·10	·14	·45
Gold 1	56	·040	·041	·63
Gold 2	100	·0035	·0036	·80
Gold 3	24	·41	·41	·05
Gold 4	35	·20	·20	·19
Silver 1 (blue) ...	36	·058	·046	·78
Silver 2 (grey) ...	39·5	·058	·055	·60
Silver 3 (grey) ...	29	·25	·42	·40
Silver 4 (blue) ...	59·7	·0022	·0019	·95
Silver 5 (grey) ...	27·3	·31	·43	·24

The thickness is in millionths of a millimetre, *i. e.* is in terms of the milli-mikrom * called by microscopists $\mu\mu$.

The films were on glass, and the absorption of the glass was allowed for by control experiments.

* In spelling the word "mikrom" thus, I am following Lord Kelvin, who wishes to introduce two units—one, the ordinary mikrometre (the mikrom) or $\frac{1}{1000}$ of a millimetre, called by microscopists μ; the other, a unit of time to be called a "michron," being the time required for light to traverse the above small distance. This suggestion of Lord Kelvin's commends itself to me; at any rate I take the opportunity of calling the attention of Physicists to it.

It is to be understood that of the whole incident light the proportion reflected is first subtracted, and the residue is then called 1 in order to reckon the fraction transmitted of that which enters the metal, it being understood that the residue which is not transmitted (say ·68 or ·63 in the case of platinum) is absorbed. It may be that more and better work has been done on the opacity of metals than this: at any rate there seems to me room for it. I do not quote these figures with a strong feeling of confidence in their accuracy. They are to be found in Wied. *Ann.* vol. xxxv. p. 57.

Appendix III. (*Added later.*)

The parenthetical statement near the bottom of page 372, concerning the first clear statement of the two circuital laws by Mr. Heaviside, represented subjectively the truth for me and for some others; but it should have been more carefully guarded if it was to be taken as representing objective truth: for I am referred to Maxwell, Philosophical Transactions, 1868 ("Collected Papers," vol. ii. p. 138), where the statement is very clear, and also to Rayleigh, Philosophical Magazine, August 1881, equations 8 and 9. I regret my inadequate acquaintance with scientific history.

XXXVIII. *On certain Diffraction Fringes as applied to Micrometric Observations.* *By* L. N. G. FILON, *M.A., Demonstrator in Applied Mathematics and Fellow of University College, London**.

1. THE following paper is largely criticism and extension of Mr. A. A. Michelson's memoir "On the Application of Interference Methods to Astronomical Observations," published in the Phil. Mag. vol. xxx. p. 256, March 1891.

Light from a distant source is allowed to pass through two thin parallel slits. The rays are then focussed on a screen (or the retina of the eye) and interference-fringes are seen. If the distant source be really double, or extended, the fringes will disappear for certain values of the distance between the slits. This distance depends on the angle subtended by the two components of the double source or the diameter of the extended source.

Mr. Michelson, however, in obtaining his results treated the breadth of the slits as small compared with the wave-length of light and their length as infinite. This seems unjustifiable *à priori*. The present investigation takes the dimensions of the slits into account.

2. Suppose we have an aperture or diaphragm of any shape in a screen placed just in front of the object-glass of a telescope (fig. 1).

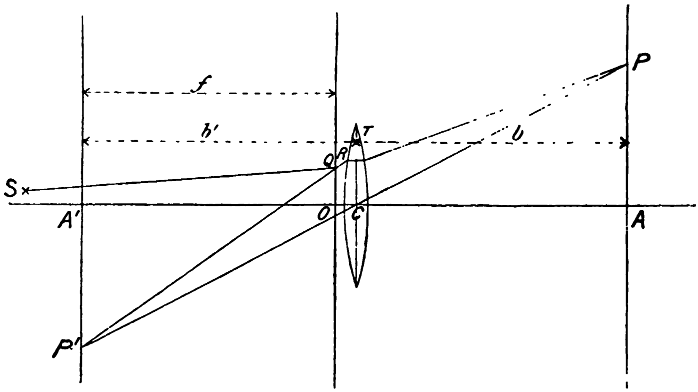
Let the axis of the telescope be the axis of z. Let the axes of x and y be taken in the plane of the diaphragm OQ perpendicular to and in the plane of the paper respectively. Let S be a source of light whose coordinates are U, V, W. Let Q be any point in the diaphragm whose coordinates are (x, y). Let AP be a screen perpendicular to the axis of the telescope, and let (p, q) be the coordinates of any point P on this screen. Let A′P′ be the conjugate image of the screen AP in the object-glass.

Let b = distance of centre C of lens from screen AP.
b' = " " " " image of screen AP.
f = distance of diaphragm OQ from plane A′P′.

* Read November 25, 1898.

Then if, as is usual, we break up a wave of light coming from S at the diaphragm, the secondary wave due to the disturbance at Q would have to travel along a path QRTP in order to reach a point P on the screen, being regularly refracted.

Fig. 1.



But since P′ is the geometrical image of P, all rays which converge to P (*i. e.* pass through P) after refraction, must have passed through P′ before refraction, to the order of our approximation.

Hence the ray through Q which is to reach P must be P′Q.

Moreover, P and P′ being conjugate images the change of phase of a wave travelling from P′ to P is constant to the first approximation and independent of the position of Q.

Now the disturbance at P due to an element $dx\,dy$ of the diaphragm at Q is of the form

$$\frac{A\,dx\,dy}{b\lambda}\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}-SQ-QR-\mu\,.\,RT-TP\right),$$

where λ is the wave-length, τ is the period, A is a constant, μ is the index of refraction of the material of the lens, and b is put instead of QP outside the trigonometrical term, because the distance of the lens from the diaphragm and the inclination of the rays are supposed small.

But $\qquad P'Q+QR+\mu RT+TP=\text{constant for } P.$

Therefore the disturbance

$$=\frac{A\,dx\,dy}{b\lambda}\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}-SQ+P'Q-\text{const.}\right).$$

But

$$SQ^2=(x-U)^2+(y-V)^2+W^2$$

$$P'Q^2=(x+\frac{b'}{b}p)^2+(y+\frac{b'}{b}q)^2+f^2.$$

Now in practice x, y, p, q are small compared with b', b, f, or W ; U and V are small compared with W. Neglecting terms of order U^3/W^3, xU^2/W^3, &c., we find

$$SQ=\sqrt{U^2+V^2+W^2}-\frac{U}{W}x-\frac{V}{W}y+\frac{1}{2}\frac{x^2+y^2}{W}.$$

In like manner

$$P'Q=b'\sqrt{\frac{f^2}{b'^2}+\frac{p^2+q^2}{b^2}}+\frac{px}{b}+\frac{qy}{b}+\frac{1}{2}\frac{x^2+y^2}{b'},$$

remembering that f is very nearly equal to b' because the diaphragm is very close to the lens.

Hence the difference of retardation measured by length in air

$$P'Q-SQ=\text{const.}+\left(\frac{p}{b}+\frac{U}{W}\right)x+\left(\frac{q}{b}+\frac{V}{W}\right)y$$
$$+\frac{1}{2}(x^2+y^2)\left(\frac{1}{b'}-\frac{1}{W}\right).$$

If now the geometrical image of S lie on the screen AP (*i.e.* if the screen is in correct focus) $b'=W$ and the last term disappears.

If, however, the screen be out of focus $1/b'$ is not equal to $1/W$, and the term in x^2+y^2 may be comparable with the two others, if b' be not very great compared with b. Thus we see that appearances out of focus will introduce expressions of the same kind as those which occur when no lenses are used.

We will, however, only consider the case where the screen is in focus. Let $-u$ and $-v$ be the coordinates of the geometrical image of S ; then

$$u/b=U/W, \qquad v/b=V/W.$$

The difference of retardation measured by length in air is therefore of the form

$$\mathrm{P'Q}-\mathrm{SQ}=\text{const.}+\frac{p+u}{b}x+\frac{q+v}{b}y.$$

Hence the total disturbance at P (integrating over the two slits) is given by the expression

$$\mathrm{D}=\frac{\mathrm{A}}{b\lambda}\int_{a-k}^{a+k}dy\int_{-h}^{h}dx\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}+\frac{p+u}{b}x+\frac{q+v}{b}y\right)$$

$$+\frac{\mathrm{A}}{b\lambda}\int_{-a-k}^{-a+k}dy\int_{-h}^{h}dx\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}+\frac{p+u}{b}x+\frac{q+v}{b}y\right),$$

where $\mathrm{A}=$ a constant,
$2a=$ distance between centres of slits,
$2k=$ breadth of either slit,
$2h=$ length of either slit.

This, being integrated out, gives

$$\mathrm{D}=\frac{2\mathrm{A}b\lambda}{\pi^2(p+u)(q+v)}\sin\frac{2\pi t}{\tau}\cos\frac{2\pi}{\lambda}\frac{q+v}{b}a\sin\frac{2\pi}{\lambda}\frac{q+v}{b}k\sin\frac{2\pi}{\lambda}\frac{p+u}{b}h,$$

whence the intensity of light

$$\iota=\frac{4\mathrm{A}^2b^2\lambda^2}{\pi^4(p+u)^2(q+v)^2}\cos^2\frac{2\pi a}{b\lambda}(q+v)\sin^2\frac{2\pi k}{b\lambda}(q+v)\sin^2\frac{2\pi h}{b\lambda}(p+u).$$

This may be written

$$\frac{64\mathrm{A}^2k^2h^2}{b^2\lambda^2}\cos^2\frac{2\pi a}{b\lambda}(q+v)\frac{\sin^2\frac{2\pi k}{b\lambda}(q+v)\sin^2\frac{2\pi h}{b\lambda}(p+u)}{\left(\frac{2\pi k}{b\lambda}(q+v)\right)^2\left(\frac{2\pi h}{b\lambda}(p+u)\right)^2}.$$

This gives fringes parallel to x and y: k being very small compared with a, the quick variation term in v is

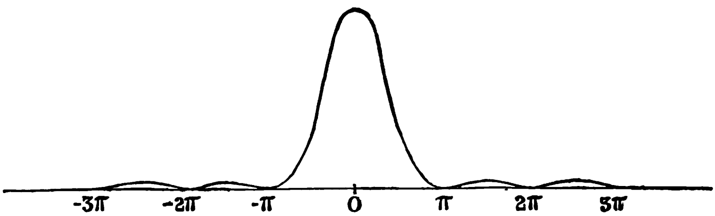
$$\cos^2\frac{2\pi a}{b\lambda}(q+v).$$

Consider the other two factors, namely:

$$\frac{\sin^2 \frac{2\pi k}{b\lambda}(q+v)}{\left\{\frac{2\pi k}{b\lambda}(q+v)\right\}^2} \quad \text{and} \quad \frac{\sin^2 \frac{2\pi h}{b\lambda}(p+u)}{\left\{\frac{2\pi h}{b\lambda}(p+u)\right\}^2}.$$

If we draw the curve $y=\frac{\sin^2 x}{x^2}$ (see fig. 2), we see that these factors are only sensible, and therefore their product is only sensible, for values of $p+u$ and $q+v$ which are numerically less than $b\lambda/2h$ and $b\lambda/2k$ respectively.

Fig. 2.



Hence the intensity becomes very small outside a rectangle whose centre is the geometrical image and whose vertical and horizontal sides are $b\lambda/k$ and $b\lambda/h$ respectively.

This rectangle I shall refer to as the "visible" rectangle of the source.

Inside this rectangle are a number of fringes, the dark lines being given by

$$q+v=\frac{2n+1}{4a}b\lambda$$

and the bright ones by $q+v=nb\lambda/2a$.

The successive maximum and minimum intensities do not vary with a. Hence, what Mr. Michelson calls the measure of visibility of the fringes, namely the quantity

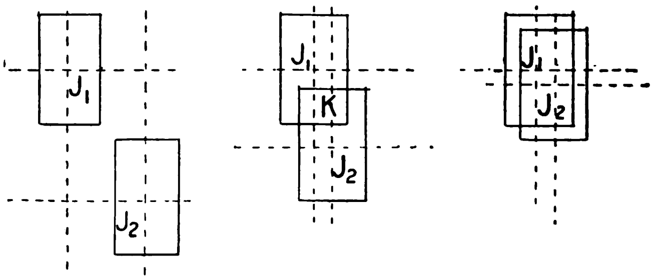
$$\frac{I_1-I_2}{I_1+I_2},$$

where I_1, I_2 are successive maximum and minimum intensities, does not vary with the distance between the slits. The only effect of varying the latter is to make the fringes close up or

open out. Hence for a point-source of light the fringes cannot be made to practically disappear.

3. Consider now two point-sources of light whose geometrical images are J_1, J_2, and draw their visible rectangles (fig. 3).

Fig. 3.



To get the resultant intensity we have to add the intensities at every point due to each source separately.

Then it may be easily seen that the following are the phenomena observed in the three cases shown in fig. 3:—

(1) The two sets of fringes distinct. Consequently no motion of the slits can destroy the fringes. In this case, however, the eye can at once distinguish between the two sources and Michelson's method is unnecessary.

(2) Partial superposition: the greatest effect is round the point K, where the intensities due to the two sources are very nearly equal. If $v'-v$ be the distance between J_1 and J_2 measured perpendicularly to the slits, so that $(v'-v)/b$ is the difference of altitude of the two stars when the slits are horizontal, then over the common area the fringe system is (a) intensified if $v'-v$ be an even multiple of $b\lambda/4a$, (b) weakened, or even destroyed, if $v'-v$ be an odd multiple of $b\lambda/4a$. For in case (a) the maxima of one system are superposed upon the maxima of the other, while in case (b) the maxima of the one are superposed upon the minima of the other. This common area, however, will contain only comparatively faint fringes, the more distinct ones round the centres remaining unaffected. We may suppose case (2) to occur whenever the centre of either rectangle lies outside the other, *i. e.* whenever $v'-v > b\lambda/2k$, $u'-u > b\lambda/2h$, $u'-u$ being the horizontal distance between J_1 and J_2.

(3) Almost complete superposition of the visible rectangles. The fringes of high intensity are now affected. These are destroyed or weakened whenever a is an odd multiple of $b\lambda/4(v'-v)$, provided that the intensity of one source be not small compared with that of the other.

Case (3) may be taken to occur when $v'-v<b\lambda/2k$ and $u'-u<b\lambda/2h$.

The smallest value of a for which the fringes disappear is $b\lambda/4(v'-v)$.

If $v'-v$ be very small, this may give a large value of a.

Now a double star ceases to be resolved by a telescope of aperture $2r$ if $(v'-v)/b=\lambda/2r$, and when this relation holds the smallest value of a for which the fringes disappear is not less than $r/2$, which is the greatest separation of the slits which can conveniently be used. Hence the method ceases to be available precisely at the moment when it is most needed.

4. Mr. Michelson, in the paper quoted above, noticed this difficulty, and described an apparatus by means of which the effective aperture of the telescope could be indefinitely increased. He has not shown, however, that the expression for the disturbance remains of the same form, to the order of approximation taken, and he has made no attempt to work out the results when the slit is taken of finite width, as it should be.

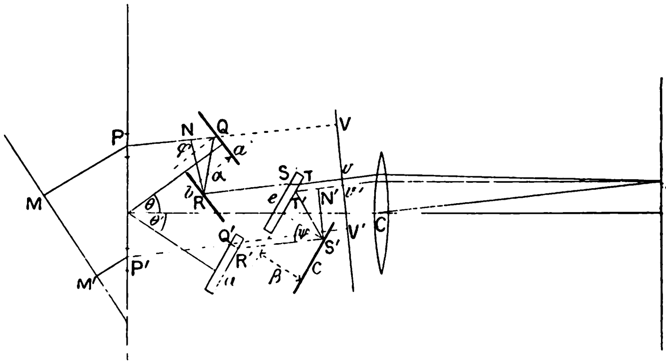
In his paper Mr. Michelson describes two kinds of apparatus. I shall confine my attention to the second one, as being somewhat more symmetrical.

So far as I can gather from Mr. Michelson's description, the instrument consists primarily of a system of three mirrors a, b, c and two strips of glass e, d (fig. 4). The mirrors a and b are parallel, and c, d, e are parallel. Light from a point P in one slit is reflected at Q and R by the mirrors a and b, is refracted through the strip e, and finally emerges parallel to its original direction as T U. Light from a point P′ in the other slit is refracted through the strip d, and reflected at S′ and T′ by the strips e and c.

I may notice in passing that the strip e should be half silvered, but *not at the back*, for if the ray S′T′ is allowed to penetrate inside the strip and emerge after two refractions

and one reflexion, not only is a change of phase introduced, owing to the path in the glass, which complicates the analysis, but the conditions of reflexion, which should be the same for all four mirrors, are altered, and this changes the intensities of the two streams. We shall see afterwards that this silvering can be done without impeding the passage of the transmitted stream, as it will turn out that the two streams must be kept separate.

Fig. 4.



Suppose then that a plane wave of light whose front is M M′ is incident upon the diaphragm. Let us break the wave up, as is usual, in the plane of the diaphragm. Let Z be a point on the screen whose cordinates are (p, q) at which the intensity of light is required.

Then if C be the centre of the object-glass, the direction in which rays T U, T′ U′ must proceed in order to converge to Z after refraction is parallel to C Z.

Hence PQ, RS, TU, P′Q′, R′S′, T′U′ are all parallel to CZ, and the direction-cosines of CZ are

$$\frac{p}{\sqrt{p^2+q^2+b^2}}, \quad \frac{q}{\sqrt{p^2+q^2+b}}, \quad \frac{b}{\sqrt{p^2+q^2+b^2}}.$$

I shall assume that the strips e and d are cut from the same plate and are of equal thickness. This will sensibly simplify the analysis, though, as I think, it would not materially influence the appearances if the strips were unequal.

If, however, we suppose them equal, we may neglect the

presence of strips, as far as refraction is concerned, since clearly the retardation introduced is the same for all parallel rays.

If now UU′ be a plane perpendicular to CZ, then, since we know that rays parallel to TU, T′U′ converge to a focus at Z, the only parts of the paths of the rays which can introduce a difference of phase are

$$\text{MP}+\text{PQ}+\text{QR}+\text{RS}+\text{TU} \text{ for one stream,}$$
$$\text{M}'\text{P}'+\text{P}'\text{Q}'+\text{R}'\text{S}'+\text{S}'\text{T}'+\text{T}'\text{U}' \text{ for the other stream.}$$

Produce PQ, P′Q′ to meet it in V and V′ and let N, N′ be the feet of the perpendiculars from R and S′ on PQ, T′U′ respectively.

Thus we may take the change of phase as due to the retardation

$$(\text{MP}+\text{PV})+(\text{NQ}+\text{QR})$$

for diffraction at one slit, and to the retardation

$$(\text{M}'\text{P}'+\text{P}'\text{V}')+(\text{S}'\text{T}'+\text{T}'\text{N}')$$

for rays proceeding from the other slit.

The terms in the first brackets give us the expression which we had before, viz. :—

$$-\left(\frac{p+u}{b}\right)x-\left(\frac{q+v}{b}\right)y+\text{const.}$$

As to the other terms

$$\text{NQ}+\text{QR}=\text{QR}\,(1+\cos 2\phi)=\alpha\frac{1+\cos 2\phi}{\cos\phi}=2\alpha\cos\phi,$$

where α is the distance between the mirrors a, b and ϕ is the angle of incidence of any ray on these mirrors.

Similarly $\text{S}'\text{T}'+\text{T}'\text{N}'=2\beta\cos\psi$, where β is the distance between e and c and ψ the angle of incidence of any ray upon e and c.

Now if the mirrors a and b are inclined to the plane of the diaphragm at an angle θ, c, d, e at an angle $(-\theta')$, then

$$\cos\phi=\frac{q\sin\theta+b\cos\theta}{\sqrt{p^2+q^2+b^2}},$$

$$\cos\psi=\frac{-q\sin\theta'+b\cos\theta'}{\sqrt{p^2+q^2+b^2}}.$$

To find the disturbance at Z we have

$$\int_{a-k}^{a+k} dy \int_{-h}^{+h} dx \frac{A}{b\lambda} \sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau} + \frac{p+u}{b}x + \frac{q+v}{b}y - 2\alpha\cos\phi\right)$$

$$+ \int_{-a-k}^{-a+k} dy \int_{-h}^{+h} dx \frac{A}{b\lambda} \sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau} + \frac{p+u}{b}x + \frac{q+v}{b}y - 2\beta\cos\psi\right),$$

which, on being integrated, gives

$$\frac{2Ab\lambda}{\pi^2(p+u)(q+v)} \sin\frac{2\pi}{\lambda}\frac{q+v}{b}k \sin\frac{2\pi}{\lambda}\frac{p+u}{b}h \sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau} - \epsilon\right)\cos\frac{2\pi\gamma}{\lambda},$$

where $-\epsilon-\gamma = -\frac{v+q}{b}a - 2\beta\cos\psi, \qquad -\epsilon+\gamma = \frac{v+q}{b}a - 2\alpha\cos\phi,$

whence $\qquad \epsilon = \alpha\cos\phi + \beta\cos\psi,$

$$\gamma = \frac{v+q}{b}a + \beta\cos\psi - \alpha\cos\phi.$$

Hence the intensity I of light at (p, q) is

$$\frac{4A^2b^2\lambda^2}{\pi^4(p+u)^2(q+v)^2} \sin^2\frac{2\pi}{\lambda}\frac{q+v}{b}v \sin^2\frac{2\pi}{\lambda}\frac{p+u}{b}h \cos^2\frac{2\pi\gamma}{\lambda}$$

where $\gamma = \frac{v+q}{b}a + \frac{\beta\cos\theta' - \alpha\cos\theta}{\sqrt{p^2+q^2+b^2}}b - \frac{(\beta\sin\theta' + \alpha\sin\theta)}{\sqrt{p^2+q^2+b^2}}q.$

In the last term we may put $\frac{q}{\sqrt{p^2+q^2+b^2}} = \frac{q}{b}$, for if we went to a higher approximation, we should introduce *cubes* of p/b, q/b which we have hitherto neglected.

If, further, we make $\beta\cos\theta' = \alpha\cos\theta$, which can always be managed without difficulty, the second term, which would contain squares on expansion, disappears and we have

$$\gamma = \frac{(v+q)a - (\beta\sin\theta' + \alpha\sin\theta)q}{b}$$

$$= \{va + q(a - (\beta\sin\theta' + \alpha\sin\theta))\}/b.$$

This gives fringes of breadth $b\lambda/2(a - (\beta\sin\theta' + \alpha\sin\theta))$. These may be reckoned from the bright fringe $\gamma = 0$; *i. e.*

$$q_0 = \frac{-va}{a - (\beta\sin\theta' + \alpha\sin\theta)}.$$

The visibility of the fringes for a single source will, as before, not be affected by changing a: for a second source the origin of the fringes is given by

$$q_0' = -v'a/(a - \{\beta \sin \theta' + \alpha \sin \theta\}).$$

and if the visible rectangles overlap, there will be a sensible diminution of the fringe appearance whenever

$$q_0 - q_0' = (n+\tfrac{1}{2})b\lambda/2\{a - (\beta \sin \theta' + \alpha \sin \theta)\}$$

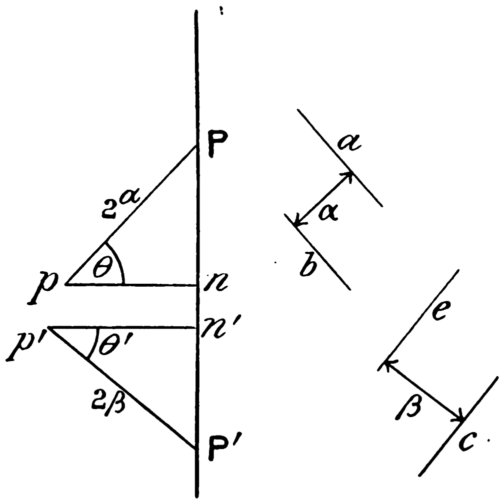
where n is an integer;

i.e. $v' - v =$ an odd multiple of $b\lambda/4a$,

the condition previously found.

One further point should be noticed : if a be very nearly equal to $\beta \sin \theta' + \alpha \sin \theta$ the fringes become too broad to be observed, whatever the source may be.

Fig. 5.



To see the physical meaning of this condition, and also of the condition $\beta \cos \theta' = \alpha \cos \theta$, we notice that a point source of light P at the centre of one of the slits appears after reflexion at the two mirrors a, b, to be at p, where Pp is equal to twice the distance between the mirrors and is perpendicular to their plane (fig. 5). Hence the double reflexion removes the image of the slit a distance $2\alpha \cos \theta$ behind the diaphragm and $2\alpha \sin \theta$ closer to the centre. In the same way the image of

the other slit is brought $2\beta \cos \theta'$ behind the diaphragm and $2\beta \sin \theta'$ nearer the centre.

Our condition $\beta \cos \theta' = \alpha \cos \theta$ therefore means that the images of the two slits must be in the same plane parallel to the plane of the diaphragm itself, and our second condition shows that they must be some distance apart.

To find the minimum of this distance, remember that the fringes will be invisible if the distance between successive maxima exceeds the vertical dimension of the visible rectangle: in other words, if

$$b\lambda/2(a-(\beta \sin \theta' + \alpha \sin \theta)) > b\lambda/k,$$

or distance in question $< k$,

which means that the centre of the image of either slit must be outside the other. These two points must be carefully borne in mind in adjusting the instruments.

When this, however, is done, we see that Michelson's assertions are confirmed, and that when we increase the aperture of the telescope in this way, the results obtained are of the same character as when the slits are placed directly in front of the object-glass.

5. Let us now proceed to consider an extended source, which we shall suppose for simplicity to be of uniform intensity.

The intensity at a point (p, q) on the screen will be of the form

$$I = A^2 \frac{64h^2k^2}{b^2\lambda^2} \iint \left\{ \frac{\sin \dfrac{2\pi k(q+v)}{b\lambda}}{\left(\dfrac{2\pi k(q+v)}{b\lambda}\right)} \cdot \frac{\sin \dfrac{2\pi h(p+u)}{b\lambda}}{\left(\dfrac{2\pi h(p+u)}{b\lambda}\right)} \right\}^2 \cos^2 \frac{2\pi a(q+v)}{v\lambda} \, du \, dv,$$

the integral being taken all over the geometrical image of the extended source.

We have now three cases to consider.

(a) When the angular dimensions of the source are large compared with λ/h.

(b) When the angular dimensions of the source are small compared with λ/h.

(c) When the angular dimensions of the source are neither large nor small compared with λ/h.

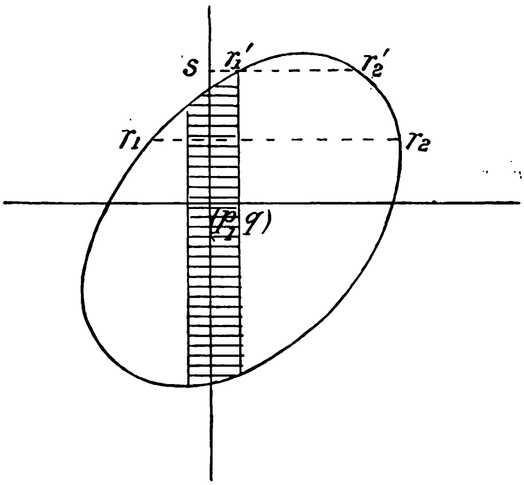
Let us begin with case (a). Then, if we consider a point inside the geometrical image, the two limits for u will be very large, except where the vertical through the point cuts the image ; the quantity

$$\left\{\frac{\sin\frac{2\pi h(p+u)}{b\lambda}}{\frac{2\pi h(p+u)}{b\lambda}}\right\}^2$$

being insensible for all points outside a thin strip (shaded in the figure) having for its central line the line through p, q perpendicular to the slits.

We may therefore, in integrating with regard to u, replace the limits by $\pm\infty$, and then integrate with regard to v along the chord of the image perpendicular to the slits.

Fig. 6.



Hence, remembering that

$$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi,$$

it follows that

$$\mathrm{I}=\frac{32\mathrm{A}^2hk^2}{b\lambda}\int\left\{\frac{\sin\frac{2\pi k}{b\lambda}(q+v)}{\frac{2\pi k}{b\lambda}(q+v)}\right\}^2\cos^2\frac{2\pi a}{b\lambda}(q+v)\,dv.$$

Now if the angular dimensions of the source of light be large compared with λ/k, the limits of integration with regard to v may be made infinite. In this case the intensity I

$$= \frac{8A^2hk^2}{b\lambda}\int_{-\infty}^{\infty} dv \left\{\sin^2\frac{2\pi}{b\lambda}(a+k)(q+v)+\sin^2\frac{2\pi}{b\lambda}(a-k)(q+v)\right.$$

$$\left.-2\sin^2\frac{2\pi a}{b\lambda}(q+v)+2\sin^2\frac{2\pi k}{b\lambda}(q+v)\right\} \div \left(\frac{2\pi k(q+v)}{b\lambda}\right)^2$$

$$= \frac{4A^2h}{\pi}\left[\{(a+k)+(a-k)-2a+2k\}\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}dx\right]$$

$$= 8A^2hk = \text{constant.}$$

This result shows us that if the dimensions of the source exceed a certain limit, no diffraction-fringes exist at all, at least near the centre of the image. Next let the angular dimensions of the source be less than $\frac{\lambda}{12k}$, then throughout the integration $\frac{2\pi k}{b\lambda}(q+v)$ is less than $\pi/6$ numerically.

But

$$\frac{\sin^2\frac{\pi}{6}}{\pi^2/36} = \frac{9}{\pi^2},$$

and differs but little from unity.

We may therefore in this case write

$$\frac{\sin^2\frac{2\pi k}{b\lambda}(q+v)}{\left(\frac{2\pi k}{b\lambda}(q+v)\right)^2} = 1$$

throughout the range of integration.

If now the limits be v_1 and v_2 we have

$$\mathrm{I} = \frac{32A^2hk^2}{b\lambda}\int_{v_2}^{v_1}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos\frac{4\pi a(q+v)}{b\lambda}\right)dv$$

$$= \frac{4A^2hk^2}{\pi a}\int_{v_2}^{v_1}\left(1+\cos\frac{4\pi a(q+v)}{b\lambda}\right)d\left(\frac{4\pi a(q+v)}{b\lambda}\right)$$

$$= \frac{4A^2hk^2}{\pi a}\left\{\frac{4\pi a}{b\lambda}(v_1-v_2)+\sin\frac{4\pi a(q+v_1)}{b\lambda}-\sin\frac{4\pi a(q+v_2)}{b\lambda}\right\}$$

$$= \frac{4A^2hk^2}{\pi a}\left\{\frac{4\pi a(v_1-v_2)}{b\lambda}+2\sin\frac{2\pi a(v_1-v_2)}{b\lambda}\cos\frac{4\pi a(q+\frac{1}{2}(v_1+v_2))}{b\lambda}\right\}.$$

Let $2c$=length of chord through the point perpendicular to the direction of the slits, then

$$2c=v_1-v_2,$$

and let v_0=coordinate of the mid-point of this chord. Then

$$\mathrm{I}=\frac{8\mathrm{A}^2hk^2}{\pi a}\left\{\frac{4\pi ac}{b\lambda}+\sin\frac{4\pi ac}{b\lambda}\cos\frac{4\pi a(q+v_0)}{b\lambda}\right\}.$$

The fringes therefore disappear when

$$\frac{4\pi ac}{b\lambda}=s\pi.$$

Their visibility is

$$\sin\frac{4\pi ac}{b\lambda}\bigg/\frac{4\pi ac}{b\lambda},$$

and is a maximum when

$$\frac{4\pi ac}{b\lambda}=\tan\frac{4\pi ac}{b\lambda};$$

but the most visible fringes correspond to the early maxima.

This form agrees exactly with the formula given by Mr. Michelson for a uniformly illuminated segment of a straight line perpendicular to the slits. We see, however, that, provided the conditions stated be fulfilled, it is applicable to a source of any shape.

The most general form of the fringes is given by

$$q+\tfrac{1}{2}(v_1+v_2)=\text{const.},$$

and therefore consists of lines parallel to the locus of middle points of chords at right angles to the slits. These will be straight lines in the case of a rectangular, circular, or elliptic source. Here, however, a new difficulty presents itself. For the rectangular source v_1-v_2 will be constant, whatever chord perpendicular to the slits we may select. Fringes will therefore appear and disappear as a whole.

But for a circular or elliptic source, v_1-v_2 *varies* as we pass from chord to chord. Thus the maxima will be invisible for some chords when they are most visible for others and conversely. Hence, whatever be the distance between the slits, it appears at first as if we might always expect a mottled appearance.

But in the case of a circular or elliptic source the length of

the chord varies extremely slowly *near the centre* and *there* fringes will be visible, the length of the chord being practically constant. The mottled appearance, on the other hand, will predominate as we approach the sides.

6. Consider now case (b) and let the dimensions of the source be so small that, for any point sufficiently close to the centre of the image $\frac{2\pi h(p+u)}{b\lambda}$ is a small angle throughout the range of integration.

[For points not near the centre of the image the illumination will be very small and the appearances are comparatively unimportant.]

For a point distant $<\frac{\lambda b}{24h}$ from the centre of the image, we may put, as in previous reasoning,

$$\left\{\frac{\sin\frac{2\pi k}{b\lambda}(q+v)}{\frac{2\pi k}{b\lambda}(q+v)}\;\frac{\sin\frac{2\pi h}{b\lambda}(p+u)}{\frac{2\pi h}{b\lambda}(p+u)}\right\}^2=1$$

all over the range of integration, whence

$$\mathrm{I}=\frac{32\mathrm{A}^2k^2h^2}{b^2\lambda^2}\iint\left(1+\cos\frac{4\pi a(q+v)}{b\lambda}\right)dudv$$

$$=\frac{32\mathrm{A}^2k^2h^2}{b^2\lambda^2}\left(\Omega+\cos\frac{4\pi aq}{b\lambda}\int u\cos\frac{4\pi av}{b\lambda}\,dv-\sin\frac{4\pi aq}{b\lambda}\int u\sin\frac{4\pi av}{b\lambda}\,dv\right)$$

$$=\frac{32\mathrm{A}^2k^2h}{b^2\lambda^2}\left(\Omega+\mathrm{R}\cos\left(\frac{4\pi aq}{b\lambda}+\phi\right)\right),$$

where $\quad\Omega=$ total area of the image,

$$\mathrm{R}\cos\phi=\int u\cos\frac{4\pi av}{b\lambda}\,dv,\quad \mathrm{R}\sin\phi=\int u\sin\frac{4\pi av}{b\lambda}\,dv,$$

the integrals being taken all over the image. The visibility $=\mathrm{R}/\Omega$ and therefore vanishes when R vanishes.

In the case of a circular source we find

$$\phi=0,\qquad \mathrm{R}=\text{(some non-vanishing factor)}, \mathrm{J}_1\left(\frac{4\pi ar}{b\lambda}\right),$$

where J_1 is the Bessel's function of order unity and r is the radius of the image, so that r/b is the angular radius of the source. The dark fringes are given by

$$\frac{4\pi aq}{b\lambda}=(2s+1)\pi,$$

q being measured from the centre of the source. The fringes are parallel to the slits and disappear whenever

$$J_1\left(\frac{4\pi ar}{b\lambda}\right)=0.$$

This result agrees with the one given by Michelson for any circular source. We see that it only holds provided the dimensions of the source do not exceed a certain limit.

In the case of an elliptic source $\phi=0$ also, and R is not altered by any sliding of the image parallel to the direction of the slits. Hence we may replace the oblique ellipse by one having its principal axis parallel and perpendicular to the slits, the values of the semi-axes being d and ϖ, where $d=$ length of semi-diameter of original image parallel to the slits, $\varpi=$ length of perpendicular from the centre upon the parallel tangent. We find without difficulty:

$$R=\int\cos\frac{4\pi av}{b\lambda}udv \text{ over the image}$$

$$=4\int_0^{\varpi}\frac{d}{\varpi}\sqrt{\varpi^2-v^2}\cos\frac{4\pi av}{b\lambda}dv=\frac{d\varpi}{2}\cdot\frac{b\lambda}{a\varpi}J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)$$

$$=\Omega\left(\frac{b\lambda}{2\pi a\varpi}\right)J_1\left(\frac{4\pi a\varpi}{b\lambda}\right).$$

The visibility is therefore

$$2\,J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)\Big/\left(\frac{4\pi a\varpi}{b\lambda}\right),$$

and vanishes whenever

$$J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)=0.$$

Hence we see that, for an elliptic source, if $\rho=$ length of semi-diameter perpendicular to the slits, $\varpi=$ length of perpendicular on the tangent parallel to the slits, then the fringes disappear

when $\sin\frac{4\pi a\rho}{b\lambda}=0$, if the angular dimensions are of order $\frac{1}{12}\frac{\lambda}{k}$ as indicated above, and when $J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)=0$, when the angular dimensions are less than $\frac{1}{12}\frac{\lambda}{h}$.

In the first case ρ is the quantity which determines the disappearance of the fringes, in the second case ϖ: and further, we see that the validity of the formulæ is entirely dependent on the *length* and *breadth* of the slit, neither of which is considered by Mr. Michelson.

We may notice that the best results are obtained, in the first case when h is large, in the second case when h is small.

7. It remains to consider the intermediate case (c). This does not perhaps present so much interest as the other two; the first will generally correspond to the case of a planet, the second to that of a star, in astronomical observations.

In dealing with case (c) we shall suppose the angular dimensions to be small, with regard to λ/k, but not with regard to λ/h. We may then write

$$I=\frac{64A^2h^2k^2}{b^2\lambda^2}\iint\left\{\frac{\sin\frac{2\pi k(q+v)}{b\lambda}}{\frac{2\pi k(q+v)}{b\lambda}}\cdot\frac{\sin\frac{2\pi h(p+u)}{b\lambda}}{\frac{2\pi h(p+u)}{b\lambda}}\right\}^2\cos^2\frac{2\pi a(q+v)}{b\lambda}dudv$$

$$=\frac{32A^2h^2k^2}{b^2\lambda^2}\iint\frac{\sin^2\frac{2\pi hu}{b\lambda}}{\left(\frac{2\pi hu}{b\lambda}\right)^2}\left(1+\cos\frac{4\pi a(q+v)}{b\lambda}\right)dudv,$$

if we only consider the appearances along the line $p=0$, taken to pass through the centre of the image, which is assumed circular or elliptic.*

Denoting $\frac{2\pi h}{b\lambda}$ by μ and expanding, we get

$$\frac{\sin^2\frac{2\pi hu}{b\lambda}}{\left(\frac{2\pi hu}{b\lambda}\right)^2}=1-\tfrac{1}{2}\cdot\frac{2^2\mu^2u^2}{3!}+\ \ldots\ +\frac{(-1)^{r-1}(2\mu u)^{2(r-1)}}{r\,.\,(2r-1)!}+\ldots,$$

* In the case of the elliptic source, the line is not, strictly speaking, $p=0$, but is the diameter conjugate to the direction of the slits, and the same formula holds.

whence

$$I=\frac{32A^2h^2k^2}{b^2\lambda^2}\left[K+\int\left(u-\frac{1}{2.3}\frac{(2\mu)^2u^3}{3!}+\ldots+\frac{(-1)^{r-1}(2\mu)^{2(r-1)}}{r(2r-1)}\frac{u^{2r-1}}{(2r-1)!}+\ldots\right)\times\cos\frac{4\pi a(q+v)}{b\lambda}dv\right]$$

where

$$K=\iint\frac{\sin^2\mu u}{\mu^2u^2}dudv$$

taken all over the source, and is essentially positive and independent of a and q.

Now

$$\int u^{2r-1}\sin\frac{4\pi a(v)}{b\lambda}dv=0$$

for a circle or ellipse.

To find the cosine integral, C, we have, d and ϖ having the same meaning as before,

$$C=4\int_0^{\varpi}\frac{d^{2r-1}}{\varpi^{2r-1}}(\varpi^2-v^2)^{r-\frac{1}{2}}\cos\frac{4\pi av}{b\lambda}dv.$$

Put $v=\varpi\cos\theta$.

$$C=4d^{2r-1}\varpi\int_0^{\frac{\pi}{2}}\sin^{2r}\theta\cos\left(\frac{4\pi a\varpi}{b\lambda}\cos\theta\right)d\theta$$

$$=\frac{2d^{2r-1}\varpi 2^r\sqrt{\pi}\Gamma(r+\frac{1}{2})}{\left(\frac{4\pi a\varpi}{b\lambda}\right)^r}J_r\left(\frac{4\pi a\varpi}{b\pi}\right)$$

$$=\frac{2d^{2r-1}\varpi\pi(2r-1)!}{2^{r-1}(r-1)!}\frac{J_r\left(\frac{4\pi a\varpi}{b\lambda}\right)}{\left(\frac{4\pi a\varpi}{b\lambda}\right)^r}.$$

Hence:

$$I=\frac{32A^2h^2k^2}{b^2\lambda^2}\left\{K+\cos\frac{4\pi aq}{b\lambda}\left(\sum_{r=1}^{r=\infty}\frac{2\pi d^{2r-1}\varpi(-1)^{r-1}}{\left(\frac{4\pi a\varpi}{b\lambda}\right)^r}\frac{\left(\frac{4\pi h}{b\lambda}\right)^{2r-2}}{r(2r-1)}\frac{J_r\left(\frac{4\pi a\varpi}{b\lambda}\right)}{2^{r-1}(r-1)!}\right)\right\}$$

$$=\frac{32A^2h^2k^2}{b^2\lambda^2}\left[K+2\cos\frac{4\pi aq}{b\lambda}\frac{\pi\varpi d}{\left(\frac{4\pi a\varpi}{b\lambda}\right)}\sum_{r=1}^{r=\infty}(-1)^{r-1}\left\{\frac{\left(\frac{4\pi hd}{b\lambda}\right)^2}{\frac{8\pi a\varpi}{b\lambda}}\right\}^{r-1}\frac{J_r\left(\frac{4\pi a\varpi}{b\lambda}\right)}{r!(2r-1)}\right].$$

Denote by Ω the total area of the image and by e the ratio

$$\left(\frac{4\pi hd}{b\lambda}\right)^2 \bigg/ \left(\frac{8\pi a\varpi}{b\lambda}\right).$$

Then the visibility

$$= \frac{2\Omega}{\mathrm{K}} \frac{1}{\frac{4\pi a\varpi}{b\lambda}} \left\{ \mathrm{J}_1\left(\frac{4\pi a\varpi}{b\lambda}\right) - \frac{e}{6}\mathrm{J}_2\left(\frac{4\pi a\varpi}{b\lambda}\right) + \dots \right.$$
$$\left. + (-1)^{r-1}\frac{e^{r-1}}{r!\,(2r-1)}\mathrm{J}_r\left(\frac{4\pi a\varpi}{b\lambda}\right) + \dots \right\}$$

The series for the visibility is absolutely convergent, because $\mathrm{J}_{n+1}(x)/\mathrm{J}_n(x)$ decreases numerically without limit as n increases without limit.

The roots of the equation

$$\mathrm{J}_1\left(\frac{4\pi a\varpi}{b\lambda}\right) - \frac{e}{6}\mathrm{J}_2\left(\frac{4\pi a\varpi}{b\lambda}\right) + \dots + (-1)^{r-1}\frac{e^{r-1}}{r!\,(2r-1)}\mathrm{J}_r\left(\frac{4\pi a\varpi}{b\lambda}\right) + \dots = 0$$

give the values of a for which the visibility vanishes.

Notice, however, that e contains d and the length of the slit, so that the values obtained for a will be functions of the horizontal diameter and of the length of the slit.

8. Besides enabling us to determine the angular distance of two point-sources and the radius of an extended source, Mr. Michelson's method allows us to detect and measure the ellipticity of a luminous disk.

Referring to the formulæ for cases (a) and (b), the visibility vanishes when

$$\sin\frac{4\pi a\rho}{b\lambda} = 0 \text{ in case } (a),$$

and when

$$\mathrm{J}_1\left(\frac{4\pi a\varpi}{b\lambda}\right) = 0 \text{ in case } (b).$$

In either of these cases, if we rotate the slits about the axis of the telescope, without altering a, then if the source is elliptic, ρ and ϖ will vary, and the visibility of the fringes will vary.

Now suppose for a given position of the slits we vary a until the visibility$=0$ for that position, and then rotate the slits and note the different inclinations for which it vanishes.

It will certainly vanish once again before a complete half-

turn has been made, namely, when the slits make an angle with the direction of either axis of the ellipse equal to that which they made at first, but on the other side of the axis.

It may vanish more than once, but since the inclinations for which it vanishes are symmetrical with regard to the axes of the ellipse, there will usually be no difficulty in determining the *directions* of the axes.

Their *lengths* can then be determined by two observations of the disappearance of the fringes, one for each of the two positions of the slits which are perpendicular to an axis.

It must, however, be noticed that the accuracy of this method for measuring ellipticity decreases with the size of the source, inasmuch as the quantity which causes the alteration in the fringes is the *difference*, not the *ratio* of the semi-axes.

To get some idea of the sensitiveness of the method, let us estimate roughly the amount of ellipticity which could be detected in a disk of angular semi-diameter 10″, taking the mean wave-length of light $\cdot 5 \times 10^{-4}$ cms.

The visibility vanishes when $\sin\frac{4\pi a\rho}{b\lambda} = 0$, and will be quite sensible when $\sin\frac{4\pi a\rho}{b\lambda} = \frac{1}{2}$, say. Hence in order that we may be able to note a sensible difference of visibility in the fringes, we must have

$$\frac{4\pi a}{\lambda}\left(\frac{\rho_1 - \rho_2}{b}\right) = \frac{\pi}{6} \text{ at least;}$$

or

$$\frac{\rho_1 - \rho_2}{b} = \frac{1}{2}\,\frac{1}{10^6}$$

if a be a little above 4 cms.

∴ difference of angular semi-axes $= \cdot 01$ (semi-diameter) *q.p.*, or the amount of ellipticity which can be detected $= \cdot 01$.

I have taken $\sin\frac{4\pi a\rho}{b\lambda} = 0$ as giving zero visibility, because this example will clearly fall under case (a).

9. Summing up the results obtained we see that:—

(1) It is possible by the observation of Michelson's interference-fringes to separate a double point-source, or detect breadth and ellipticity in a slightly extended source.

(2) But the distance between the two points, or the dimensions of the extended source, must lie within certain limits depending on the length and breadth of the slits *.

(3) The dimensions of the slits also considerably affect the general theory, the formulæ obtained not being identical with Michelson's. The law of appearance and disappearance of the fringes depends very largely on the distance between the points or the dimensions of the extended source.

Discussion.

Prof. S. P. Thompson asked what was the minimum width of slit with which the author had found it practicable to work.

In reply, the Author said that with monochromatic light the minimum width he was able to use with his telescope was about half a millimetre.

* Since the above was read, a paper has appeared in the *Comptes Rendus de l'Académie des Sciences* for Nov. 28, 1898, dealing with the modifications in Michelson's formulæ when we take into account the breadth of the slits. The author, M. Hamy, follows Michelson in not considering variations of intensity parallel to the slits. This, I think, accounts for his results not quite agreeing with mine.

XXXIX. *The Equivalent Resistance and Inductance of a Wire to an Oscillatory Discharge.* By Edwin H. Barton, *D.Sc., F.R.S.E., Senior Lecturer in Physics, University College, Nottingham* *.

In an article in the Philosophical Magazine for May 1886 †, Lord Rayleigh, whilst greatly extending Maxwell's treatment of the self-induction of cylindrical conductors, confined the discussion of alternating currents to those which followed the harmonic law with constant amplitude. The object of the present note is to slightly modify the analysis so as to include also the decaying periodic currents obtained in discharging a condenser and the case of the damped trains of high-frequency waves generated by a Hertzian oscillator and now so often dealt with experimentally. In fact, it was while recently working with the latter that the necessity of attacking this problem occurred to me.

Résumé of previous Theories.—To make this paper intelligible without repeated references to both Maxwell and Rayleigh, it may be well to explain again the notation used and sketch the line of argument followed.

The conducting wire is supposed to be a straight cylinder of radius a, the return wire being at a considerable distance. The vector potential, H, the density of the current, w, and the "electromotive force at any point" may thus be considered as functions of two variables only, viz., the time, t, and the distance, r, from the axis of the wire. The total current, C, through the section of the wire, and the total electromotive force, E, acting round the circuit, are the variables whose relation is to be found. It is assumed that

$$H=S+T_0+T_1r^2+\ \ldots\ldots\ +T_nr^{2n}, \quad . \quad . \quad . \quad (1)$$

where S, T_0, T_1, &c., are functions of the time. A relation between the T's is next established so that the subscripts are replaced by coefficients. The value of H at the surface of the

* Read January 27, 1899.

† "On the Self-Induction and Resistance of Straight Conductors."

wire is equated to AC, where A is a constant. This leads to Maxwell's equation (13) of art. 690. The magnetic permeability, μ, of the wire, which Maxwell had treated as unity, is now introduced by Lord Rayleigh, who thus obtains in place of Maxwell's (14) and (15) the following equations :—

$$\mu C = -\left(\alpha\mu\frac{dT}{dt}+\frac{2\alpha^2\mu^2}{1^2.2^2}\cdot\frac{d^2T}{dt^2}+\ldots.+\frac{n\alpha^n\mu^n}{1^2.2^2\ldots n^2}\cdot\frac{d^nT}{dt^n}+\ldots\right), \quad . \quad . \quad (2)$$

$$AC-S = T+\alpha\mu\frac{dT}{dt}+\frac{\alpha^2\mu^2}{1^2.2^2}\frac{d^2T}{dt^2}+\ldots.+\frac{\alpha^n\mu^n}{1^2.2^2\ldots n^2}\frac{d^nT}{dt^n}+\ldots.\Big), \quad (3)$$

where α, equal to l/R, represents the conductivity (for steady currents) of unit length of the wire.

By writing

$$\phi(x)=1+x+\frac{x^2}{1^2.2^2}+\ldots.+\frac{x^n}{1^2.2^2\ldots n^2}+\ldots \quad . \quad (4)$$

equations (2) and (3) are then transformed as follows :

$$\frac{dS}{dt}=A\frac{dC^*}{dt}-\phi\left(\alpha\mu\frac{d}{dt}\right)\cdot\frac{dT}{dt}, \quad . \quad . \quad . \quad (5)$$

$$C=-\alpha\phi'\left(\alpha\mu\frac{d}{dt}\right)\cdot\frac{dT}{dt}: \quad . \quad . \quad . \quad . \quad (6)$$

we have further

$$\frac{E}{l}=\frac{dS}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Lord Rayleigh then applies equations (5), (6), and (7) to sustained periodic currents following the harmonic law, where all the functions are proportional to e^{ipt}, and obtains

$$E=R'C+ipL'C, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

R' and L' denoting the effective resistance and inductance respectively to the currents in question. The values of R' and L' are expressed in the form of infinite series. For high frequencies, however, they are put also in a finite form, since, when p is very great, equation (4) reduces analytically to

$$\phi(x)=\frac{1}{2\sqrt{\pi}}\frac{e^{2\sqrt{x}}}{x^{\frac{1}{4}}}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

* AC is printed here in Phil. Mag., May 1886, p. 387 ; but appears to be a slip for $A\frac{dC}{dt}$.

so that

$$\frac{\phi(x)}{\phi'(x)} = x^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equivalent Resistance and Inductance for Oscillatory Discharges.—To effect the object of this paper we must now apply equations (5), (6), and (7) to the case of logarithmically-damped alternating currents where all the functions are proportional to $e^{(i-k)pt}$.

The value of E so obtained must then be separated into real and imaginary parts as in (8), and then, together with the imaginary quantities, must be collected a proportionate part of the real ones so as to exhibit the result in the form

$$E = R''C + (i-k)pL''C. \quad . \quad . \quad . \quad . \quad (11)$$

The quantities denoted by R″ and L″ in this equation will then represent what may be called the *equivalent* resistance and inductance of length l of the wire to the damped periodic currents under discussion. For, the operand being now $e^{(i-k)pt}$, the time differentiator produces $(i-k)p$, and not ip simply as in equation (8) for the sustained harmonic currents.

Thus (5), (6), and (7), on elimination of S, l, and $\frac{dT}{dt}$, give

$$\frac{E}{RC} = (i-k)p\alpha A + \frac{\phi(ip\alpha\mu - kp\alpha\mu)}{\phi'(ip\alpha\mu - kp\alpha\mu)}. \quad . \quad . \quad (12)$$

Now we have

$$\frac{\phi(x)}{\phi'(x)} = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{48} - \frac{x^4}{180} + \ldots; \quad . \quad (13)$$

thus

$$\frac{\phi(ip\alpha\mu - kp\alpha\mu)}{\phi'(ip\alpha\mu - kp\alpha\mu)}$$

$$= 1 - \tfrac{1}{2}kp\alpha\mu + \frac{1-k^2}{12}p^2\alpha^2\mu^2 + \frac{k(3-k^2)}{48}p^3\alpha^3\mu^3 - \frac{1-6k^2+k^4}{180}p^4\alpha^4\mu^4 \ldots$$

$$+ i\left\{\tfrac{1}{2}p\alpha\mu + \frac{k}{6}p^2\alpha^2\mu^2 - \frac{1-3k^2}{48}p^3\alpha^3\mu^3 - \frac{4k(1-k^2)}{180}p^4\alpha^4\mu^4 \ldots\right\}. \quad (14)$$

Hence, substituting (14) in (12) and collecting the terms as in (11), we find that

$$\frac{R''}{R} = 1 + \frac{1+k^2}{12}p^2\alpha^2\mu^2 + \frac{k(1+k^2)}{24}p^3\alpha^3\mu^3 - \frac{1-2k^2-3k^4}{180}p^4\alpha^4\mu^4 \ldots, \quad (15)$$

and

$$\left.\begin{aligned}\frac{L''}{R}&=\alpha A+\tfrac{1}{2}\alpha\mu+\frac{k}{6}p\alpha^2\mu^2-\frac{1-3k^2}{48}p^2\alpha^3\mu^3-\frac{k(1-k^2)}{45}p^3\alpha^4\mu^4\dots\\ \text{or}\\ L''&=l\left[A+\mu(\tfrac{1}{2}+\frac{k}{6}p\alpha\mu-\frac{1-3k^2}{48}p^2\alpha^2\mu^2-\frac{k(1-k^2)}{45}p^3\alpha^3\mu^3\dots)\right]\end{aligned}\right\}(16)$$

Putting $k=0$ in these equations and denoting by single-dashed letters the corresponding values of the resistance and inductance, we have

$$\frac{R'}{R}=1+\frac{1}{12}p^2\alpha^2\mu^2-\frac{1}{180}p^4\alpha^4\mu^4\dots, \quad . \quad . \quad (17)$$

and

$$L'=l\left[A+\mu(\tfrac{1}{2}-\frac{1}{48}p^2\alpha^2\mu^2\dots)\right], \quad . \quad . \quad (18)$$

which are Lord Rayleigh's well-known formulæ * for periodic currents of constant amplitude.

By taking the differences of the resistances and inductances with damping and without, we have at once

$$\frac{R''-R'}{R}=k^2p^2\alpha^2\mu^2+\frac{k(1-k^2)}{24}p^3\alpha^3\mu^3+\dots \quad . \quad . \quad (19)$$

and

$$L''-L=l\mu\left(\frac{k}{6}p\alpha\mu+\frac{k^2}{6}p^2\alpha^2\mu^2\dots\right). \quad . \quad . \quad (20)$$

These show that if the frequency is such that a few terms sufficiently represent the value of the series, then *both resistance and inductance are increased by the damping.*

High-Frequency Discharges.—Passing now to cases where p is very great, as in the wave-trains in or induced by a Hertzian primary oscillator, we have from equation (10),

$$\frac{\phi(ip\alpha\mu-kp\alpha\mu)}{\phi'(ip\alpha\mu-kp\alpha\mu)}=\sqrt{(i-k)p\alpha\mu}=(p\alpha\mu s)^{\frac{1}{2}}\left(i\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right), \quad (21)$$

where $s=\sqrt{1+k^2}$ and $\cot\theta=k$.

* Equations (19) and (20), p. 387, *loc. cit.*

On substituting this value of ϕ/ϕ' in equation (12) and collecting as before, we obtain the solution sought, viz.:

$$\frac{\mathrm{E}}{\mathrm{RC}}=(\alpha\mu p s^3)^{\frac{1}{2}}\cos\frac{\theta}{2}+(i-k)p\left(\alpha\mathrm{A}+\sqrt{\alpha\mu s/p}\cos\frac{\theta}{2}\right); \quad (22)$$

whence

$$\frac{\mathrm{R}''}{\mathrm{R}}=(\alpha\mu p s^3)^{\frac{1}{2}}\cos\frac{\theta}{2}, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and

$$\left.\begin{aligned}\frac{\mathrm{L}''}{\mathrm{R}}&=\alpha\mathrm{A}+(\alpha\mu s/p)^{\frac{1}{2}}\cos\frac{\theta}{2},\\ \text{or}\quad \mathrm{L}''&=l\left[\mathrm{A}+\left(\frac{\mu s}{\alpha p}\right)^{\frac{1}{2}}\cos\frac{\theta}{2}\right].\end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

Discussion of the Results for High Frequencies.

On putting $k=0$, in equations (23) and (24), to reduce to the case of sustained simple harmonic waves, $s=1$, $\theta=\frac{\pi}{2}$; whence, denoting by single dashes these special values of R″ and L″, we obtain

$$\frac{\mathrm{R}'}{\mathrm{R}}=\sqrt{\tfrac{1}{2}\alpha\mu p}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and

$$\mathrm{L}'=l\left\{\mathrm{A}+\sqrt{\frac{\mu}{2\alpha p}}\right\}, \quad . \quad . \quad . \quad . \quad (26)$$

which are Lord Rayleigh's high-frequency formulæ*.

Referring again to equations (23) and (24), we see that for a given value of p, if k varies from 0 to ∞, the factor involving s increases without limit while that involving θ increases to unity. Hence, with increasing damping, it appears that R″ and L″ each increase also, while ever the equations remain applicable. Now an infinite value of k involves zero frequency†. And a certain large, though finite, value of k would prevent the frequency being classed as "high."

* Equations (26) and (27), p. 390, *loc. cit.*

† This follows from the fact that electric currents or waves generated by an oscillatory discharge may be represented by $e^{-kpt}\cos pt$, in which kp is finite, so k is infinite only when p is zero.

Dividing equation (23) by (25) gives

$$\frac{R''}{R'}=(2s^3)^{\frac{1}{2}}\cos\frac{\theta}{2}=K \text{ say. . . . .} \quad (27)$$

Thus, for a given value of k, the ratio R''/R' is independent of the frequency of the waves. It is therefore convenient to deal with K a function of k only, rather than with R''/R which is a function of p also.

Differentiating to k, we have

$$\frac{dK}{dk}=\frac{3k\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\sqrt{2s}}, \quad \text{. . . . . .} \quad (28)$$

which is positive for all values of k from 0 to ∞, hence K increases continuously with k. For $k=0$, this becomes

$$\left(\frac{dK}{dk}\right)_{k=0}=\frac{\sin\frac{\pi}{4}}{\sqrt{2}}=\frac{1}{2}, \quad \text{. . . . .} \quad (29)$$

which assists in plotting K as a function of k.

Differentiating again, we obtain

$$\frac{d^2K}{dk^2}=\frac{1}{\sqrt{2^3s^5}}\left\{(7+3k^2)\cos\frac{\theta}{2}+2k\sin\frac{\theta}{2}\right\} \quad . \quad (30)$$

Since this expression is positive for all values of k from 0 to ∞, we see that K plotted as a function of k is a curve which is always convex to the axis of k. Thus the nature of R''/R' as a function of k is sufficiently determined.

Pairs of corresponding values of K and k for a few typical cases are shown in the accompanying table, and part of the curve coordinating them is given in fig. 1. It is not necessary to plot much of the curve, as only a small part of it can apply to any actual case. For, although k may have any positive value up to ∞, the high values of k, as already mentioned, correspond to low values of p and so exclude them from the application of the high-frequency formula.

Fig. 1.—Exhibiting graphically $K = R''/R'$ as a function of k, the damping factor.

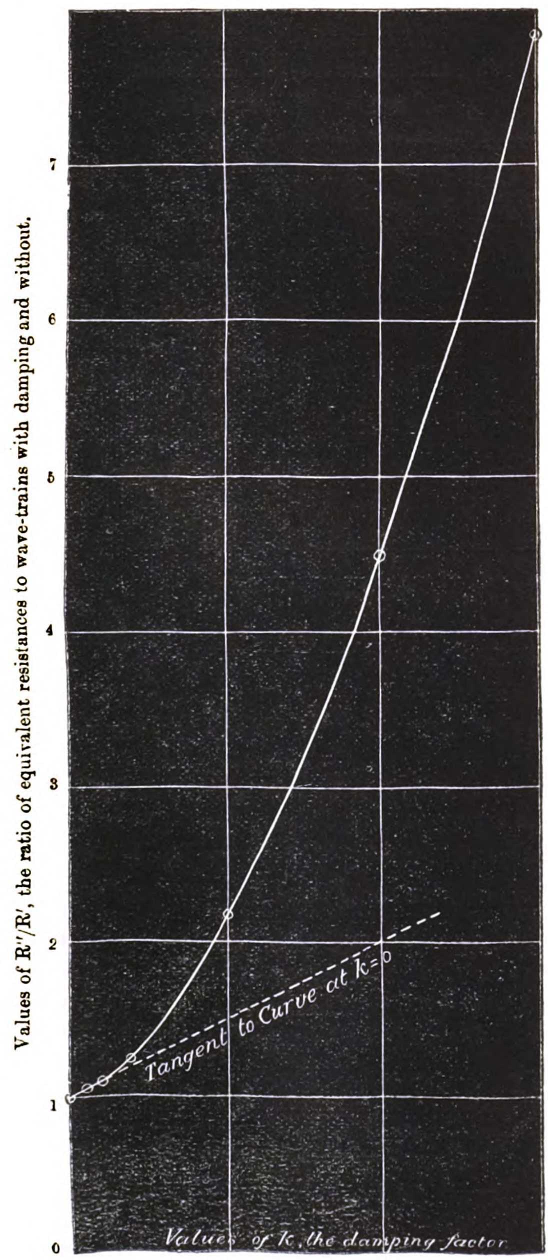


TABLE showing the values of $K=R''/R'$, the ratio of equivalent resistances to waves with damping and without.

Damping Factor, $k=\cot\theta$.	Subsidiary quantities involved.		Ratio of Resistances $K=R''/R'$.
	$\theta/2$.	$s^2=1+k^2$.	
0	45°	1	1
$\frac{1}{4\pi}=0{\cdot}0798$	42° 44′	1·006362	1·044 nearly
$\frac{3}{10\pi}=0{\cdot}0955$	42° 16′	1·00913	1·054 ,,
$\frac{1}{2\pi}=0{\cdot}1595$	40° 28′	1·02614	1·097 ,,
$\frac{1}{\pi}=0{\cdot}319$	36° 9′	1·1018	1·228 ,,
1	22° 30′	2	2·197 ,,
2	13° 17′	5	4·602 ,,
3	9° 13′	10	7·85 ,,

Figure 2 shows the form of a wave-train for which $k=1$ and $K=2{\cdot}197$. That is to say, in this extreme case where all the functions vary as $e^{(i-1)pt}$, and the wave-train passing a given point of the wire is accordingly represented by $e^{-pt}\cos pt$, then the equivalent resistance is 2·197 times that which would obtain for simple harmonic waves uniformly sustained and of the same frequency.

Figure 3 represents the form of the wave-trains generated and used in some recent experiments on attenuation*. In this case the value of k was approximately $\frac{3}{10\pi}$, or the logarithmic decrement per wave $=2\pi k=0{\cdot}6$, and the corresponding value of K, the ratio of R''/R', is 1·054. Now in the experiments just referred to the frequency was 35×10^6 per second, and R'/R became 31·6. Hence R''/R has the

* "Attenuation of Electric Waves along a Line of Negligible Leakage," Phil. Mag. Sept. 1898, pp. 296–305.

Fig. 2.—Instantaneous Form of Wave-train for $k=1$, whence $R''/R'=2{\cdot}197$.

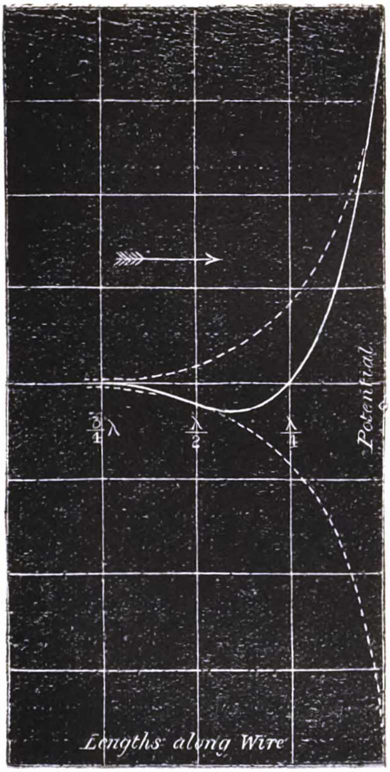
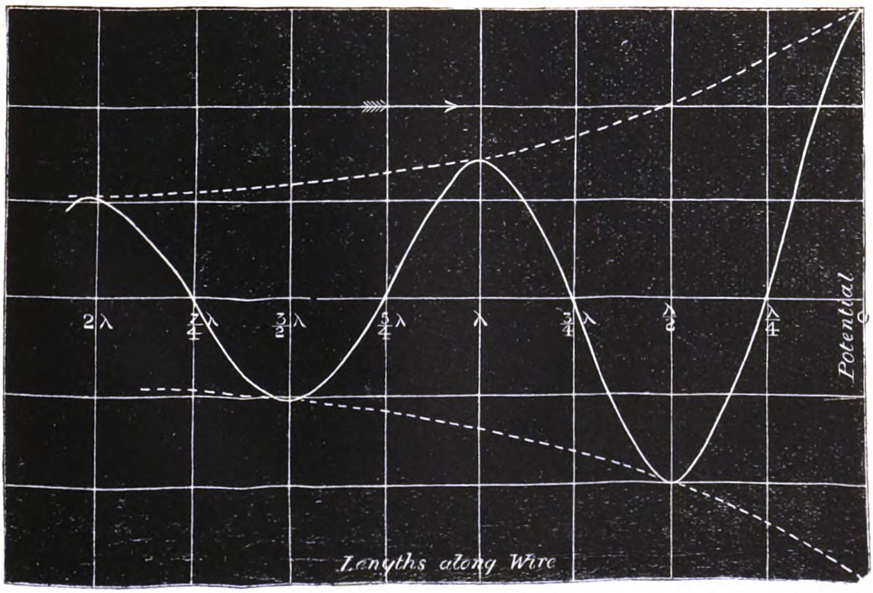


Fig. 3.—Instantaneous Form of Wave-Train for $k={\cdot}3/\pi$, whence $R''/R'=1{\cdot}054$.



value $31\cdot6\times1\cdot054=33\cdot3$ nearly. Thus, writing $e^{-R''x/2Lv}$ for the attenuator of the waves along the wires instead of * $e^{-R'x/2Lv}$ increases the index by about five and a half per cent., and so brings it by that amount nearer to the value determined experimentally.

Univ. Coll., Nottingham,
Nov. 29, 1898.

Discussion.

Mr. Oliver Heaviside (communicated).—I have in another way obtained the same number $R''/R'=1\cdot054$, showing that the effective resistance given by Lord Rayleigh's formula should be increased 5·4 per cent. when the impressed vibrations are damped to the extent Dr. Barton found in his experiments. This R'' is what should take the place of the resistance per unit length of circuit in the factor $\epsilon^{-Rx/2Lv}$, showing the attenuation in transit of waves along the circuit. This R'', however, underestimates the resistance, because the formula is not valid right up to the wave front, but is what is tended to later on. But it does not seem likely that a complete reckoning of the effect of the variable resistance of the wires will make up the deficit of theory from observation. Some other causes of increased attenuation have been suggested. In addition it is even possible that the conductivity of copper to vibrations millions per second may be less than with steady currents, and that the voltage at the beginning of the wave-train may be large enough to cause some leakage.

Both R'' and L'' go up to infinity with infinite increase in the damping. But the change in the inductance is quite insensible in the experiment. The inductance is sensibly that of the dielectric.

As regards the meaning of R'' and L'', they differ somewhat from R' and L'. When we reduce the equation of voltage $E=ZC$ to the form $E=\left(R''+L''\frac{d}{dt}\right)C$, both E and C being of the type $\epsilon^{-at}\sin(nt+\theta)$, we have the activity

* See Equation (2) p. 301, Phil. Mag. Sept. 1898.

equation

$$EC = R''C^2 + \frac{d}{dt}\tfrac{1}{2}L''C^2,$$

and also

$$EC = Q + \frac{dT}{dt},$$

where Q is the waste and T the magnetic energy. But the mean Q is not the same as the mean $R''C^2$, nor the mean T the same as the mean $\frac{1}{2}L''C^2$, save when $a=0$, or the vibrations are undamped. It is true, however (as I have investigated), that the mean $Q\epsilon^{2at}$ and $T\epsilon^{2at}$ are the same as the mean $R''C^2\epsilon^{2at}$ and $\frac{1}{2}L''C^2\epsilon^{2at}$, so that if ϵ^{2at} does not change sensibly in a period, R'' and L'' are sensibly the effective resistance and inductance in the same sense as R' and L'.

XL. *Exhibition and Description of Wehnelt's Current-Interrupter.* *By* A. A. CAMPBELL SWINTON.*

[Abstract.]

A GLASS cell contains a large cylindrical negative electrode of lead, and a small positive electrode consisting of a platinum wire about $\frac{1}{4}$ inch in length, in a solution of 1 part sulphuric acid to about 5 parts water. The platinum wire may project from the top of the shorter arm of a J-shaped ebonite tube, so that it can point upwards, immersed in the solution. Or it may be fused into a similar glass tube; but glass is apt to crack in the subsequent heating. Wehnelt's interrupter replaces the make-and-break apparatus of an induction-coil; it also replaces the ordinary condenser of that apparatus. In its present form it requires rather a strong current. The resulting spark at the secondary terminals differs in character from the ordinary spark of an induction-coil; it is almost unidirectional, and in air takes a Λ-form—bright, continuous, and inverted—somewhat like a pair of flaming swords rapidly

* Read March 10, 1899.

crossing and re-crossing one another at their points. By blowing upon the discharge it breaks up, and then more nearly resembles the customary discharge of a coil. The sound emitted by the spark has a pitch that varies with the conditions of the circuit. As the self-induction of the circuit is diminished, the spark-pitch rises; it becomes infinite when tke self-induction vanishes, *i. e.*, the Wehnelt interrupter will not work in a circuit devoid of self-induction. As the applied potential difference diminishes, the spark-pitch diminishes. In the author's experiments 25 volts was the minimum primary voltage at which the apparatus would work. The spark-pitch also varies with the length of the platinum-wire electrode in the solution. If the circuit is closed by slowly dipping this electrode into the solution, the apparatus will not work; the wire should be dipped in before closing the circuit, or at any rate immersed with considerable rapidity. After working for about a quarter of an hour the action often ceases; this fatigue effect is not due to heating of the solution, for it is not obviated by keeping the temperature constant by a water-bath. It is supposed that the oxygen generated at the platinum electrode forms a more or less insulating film which interrupts the current until absorbed by the surrounding water. The fact that oxygen is more easily absorbed than hydrogen, may explain why it is necessary to connect the platinum electrode to the positive pole of the battery or dynamo. When the platinum electrode is dipped gradually into the solution, the wire gets red-hot, and the interruptions do not take place. Again, when the apparatus stops from fatigue the platinum gets red-hot. The action is further complicated by a series of small explosions, and by the formation of a kind of intermittent electric arc at the platinum electrode. The coil exhibited was connected to the 100-volt electric light mains at Burlington House; in this case the potential difference at the terminals of the primary was 30 volts, and that across the interrupter 150 volts—a total of 180 volts, showing the effect of self-induction. For Röntgen-ray work the apparatus would be very effective, but unfortunately the sparks produce great heating, so that the terminals of the tubes are liable to be melted.

The author suggests that, as the sparks are more nearly continuous than ordinary discharges if used for producing Hertz waves, the trains of waves would follow one another at shorter intervals than those from the sparks at present employed. In fact it might eventually be possible in this way to obtain the almost absolutely continuous trains of waves that are necessary for proper syntony in wireless telegraphy.

Discussion.

The President asked whether the self-induction of the primary coil was not sufficient of itself to form the induction factor in the impedance necessary for perfect working. He would like to know how the apparatus behaved when an alternating current was used. Did the secondary coil become damaged by over-heating? Did reversal of the current assist the recovery from the fatigued condition of the apparatus? The natural period of the circuit probably depended upon its capacity and its self-induction. There must undoubtedly be some capacity at the surface of the platinum electrode in the liquid; this capacity might act with the auxiliary self-induction and the self-induction of the rest of the circuit in the orthodox way, and possibly there was automatic adjustment of resonance to the frequency of the interruptions, for instance, by variations of the capacity at the electrode. The heating effect when a wire was made to close a circuit with a liquid was discovered many years ago.

Prof. G. M. Minchin thought that the usefulness of the apparatus would be greatly increased if it could be made to work with less current. He had himself succeeded with an applied E.M.F. of 12 volts, but not with 10 volts. As a tentative experiment he had used a horizontal lead plate—with disastrous effect, for the apparatus went suddenly to pieces. Explosions were frequently obtained, but they were not attended with much real danger. In a later and safer apparatus he used a platinum wire about $\frac{3}{4}$ inch long, projecting from a glass tube, around which the lead plate was bent. There appeared to be a definite depth of immersion of this wire,

at which the apparatus worked with minimum current. In his apparatus this critical position was when half the wire was below the surface of the liquid, the other half projecting into the air. He attributed the fatigue to the presence of gas about the electrodes, for he observed that a mechanical tap to the base of the apparatus restored the working condition.

Mr. ROLLO APPLEYARD pointed out that the improved result at half immersion observed by Prof. Minchin, taken together with the phenomena described by Mr. Campbell Swinton as to the effect of dipping the electrodes into the solution, suggested that the liquid immediately around the submerged part of the wire was at some instants in the spheroidal state. The breaking-down of the spheroidal state would be facilitated by heat lost by the immersed part to the non-immersed part of the wire. The capacity for heat of the non-immersed part, and the degree of roughness or smoothness of the immersed part, would thus appear as factors in the explanation. No doubt the evolved gases were the primary cause of the interruption of current, but the wire having once become red-hot the spheroidal condition would introduce a further cause of electrical separation between the wire and the liquid.

Prof. C. V. BOYS asked whether it was the liquid or the electrodes that became fatigued. Experiments should be made to determine the effect of variations in the hydrostatic pressure around the platinum electrode.

Mr. T. H. BLAKESLEY said that the rise of potential at the terminals of the interrupter proved that the arrangement possessed capacity. Such a rise of potential could not occur without there being capacity any more than it could without self-induction.

Dr. D. K. MORRIS described experiments he had made with a Wehnelt interrupter, using a 1 kilowatt transformer with a transformation ratio of 4 to 5, intended for 10 amperes at 100 volts. The anode of the interrupter was designed to have an adjustable surface to correspond with the load on the secondary—a platinum wire at the end of a copper wire could be projected more or less through the drawn-out lower end of a glass tube containing oil. The best results with the

interrupter were obtained with about 45 volts on the primary circuit. At this pressure, an average current of 1 ampere sufficed to give 125 (alternating) volts very steadily on the secondary, as measured by an electrostatic instrument. The "no-load" loss was thus only 45 watts. The secondary could then be loaded up with lamps, provided that the exposed surface of platinum wire was proportionately increased. The energy delivered to the lamps, however, was not at any load much greater than 45 per cent. of that taken from the mains. By connecting the interrupter with a condenser of ½ microfarad capacity, the efficiency at small loads was increased to nearly 60 per cent. He had observed that the fatigue of the interrupter could be temporarily remedied by reversing the current.

Mr. C. E. S. Phillips asked whether Mr. Campbell Swinton had tried other liquids than dilute sulphuric acid. So far as his own experiments went, he had only obtained good results with that electrolyte.

Mr. Campbell Swinton, in reply, said that with the apparatus arranged in a simple circuit, an alternating current applied to the primary of an induction-coil through a Wehnelt interrupter produced only about half the effect of the corresponding direct current—apparently only the currents in one direction got through. But if two interrupters were connected in parallel circuits, it was possible so to arrange them that one took one half and the other the second half of the alternations. It might therefore be possible to design an induction-coil with two primary windings to correspond to the two interrupters, so as to give an additive effect. The ten-inch induction-coil he had used had suffered no damage from the currents employed in the experiments exhibited; though he had used as much as 20 primary amperes, there was extremely little heating of the secondary. He could not with his apparatus restore the working condition by any mechanical disturbance of the interrupter when once the fatigue effects had set in to any marked extent. He had tried other liquids in place of the dilute sulphuric acid. Whilst a saturated solution of potassic bichromate gave fair results, strong hydrochloric acid would not work at all.

The PRESIDENT said he did not altogether agree with Mr. Campbell Swinton's remarks as to the chances of being able to maintain electric oscillations, and so of improving Hertzian telegraphy, by the use of these interrupters. The rate of interruption with this apparatus was something like 1000 per second, but the vibrations corresponding to Hertz waves were of the order 100,000 per second. The wave-trains from oscillations excited by the new interrupter would still be series of damped vibrations; the amplitudes would not be maintained. It might be advantageous to have sparks following one another so rapidly, but he doubted it. For Hertzian telegraphy, the spark at the oscillator should 'crackle'; to produce the best effect, the air about the oscillator should be in a non-conducting condition.

XLI. *A new form of Amperemeter and Voltmeter with a Long Scale.* By BENJAMIN DAVIES *. (Communicated by Prof. Lodge.)

THESE measuring-instruments, which are intended for continuous currents, are of the so-called d'Arsonval type, and their chief feature is their long and uniformly divided scale. Mainly on account of these advantages, I am of opinion that the instruments may become generally useful for electrical measurement on switch-board or on the bench, and particularly in the laboratory for purposes of experiment and research.

With the exception of instruments depending on magnification, these (so far as I am aware) are the only long-range ones yet designed; a brief description of them may, therefore, not be unwelcome †.

It is evident that a long scale is a desirable addition to the best of instruments, provided the extra length can be produced without increasing the waste-power of the meter, or without reducing the deflecting moment. It is evident that this is a question of design only, and the instruments described below have a satisfactorily large deflecting moment, although the scale covers an arc varying from 210° to 230°. By sacrificing a little of the deflecting force, or by increasing the voltage drop at the terminals of the moving coil, an instrument may be constructed with a scale-length equal to an arc of 270°.

The magnetic circuit also has one distinct point in its favour, viz., its single air-gap. This air-gap may be as narrow as $\frac{1}{16}$ of an inch, reducing the demagnetizing force to a small quantity.

The permanent magnet and the magnetic circuit generally may have many forms, each having its own advantages and

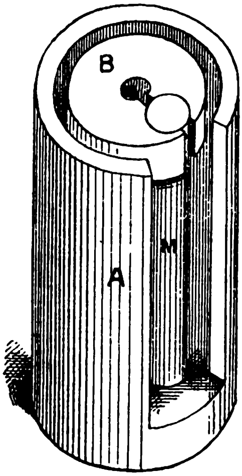
* Read February 10, 1899.

† After this paper was read I found I was by no means the first in the field as regards "long range." Professor Ayrton kindly sent me a sketch of an amperemeter by M. Carpentier, built on precisely the same principle as those mentioned in this paper. The Carpentier instrument, however, is not of good design, resulting, I should imagine, in an excessive drop of voltage at the terminals of the moving coil. The Carpentier meter, I am told, was brought out in 1889. The first form of mine was made in 1895.

disadvantages. The construction of the earliest form is shown in fig. 1. A is an iron case, either wrought or cast. B is the soft iron polepiece, which is carried and maintained in position by the permanent steel magnet M.

From a physical point of view, this combination forms a very good magnetic circuit, for the reluctance of the air-gap between polepiece and box is very small, owing to its extensive area. The relative thickness of the gap shown in the figure is much greater than it is in the actual instrument.

Fig. 1.



The form of Magnetic Circuit first employed.

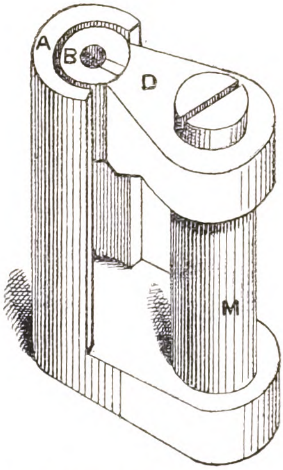
When this form was tried it was found to have some defects on practical grounds; the attachment and adjustment of the moving system became a difficult matter. Once made, however, an instrument having this form of magnetic circuit is not at all a bad one, and of all forms it offers the longest range.

The moving coil was rectangular in shape, and one side of it—forming the axis around which it rotated—passed through the central hole in B, while the opposite side moved in the narrow air-gap. This form of magnetic circuit (fig. 1) is not unlike that adopted by Mr. W. Hibbert in his apparatus for

a standard magnetic field, except that in mine the steel magnet is eccentric *.

The next form is shown in fig. 2. This is much more easily manipulated, and the difficulty of adjusting the movable system is considerably reduced. M is the steel magnet, A and B D are the two wrought-iron poles. The frame supporting the moving system is in this case mounted on the polepiece A.

Fig. 2.



Another form of Magnetic Circuit.

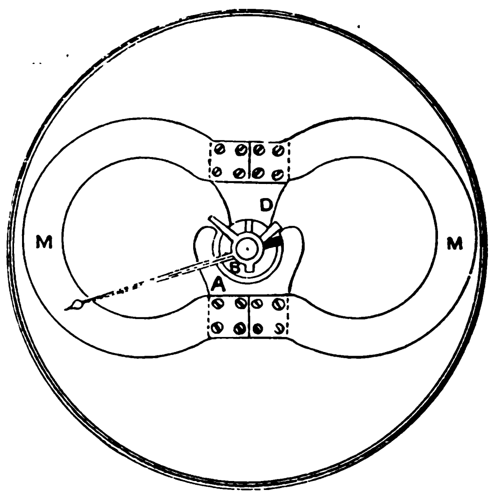
In fig. 3 we have a plan of a still better form considered from a practical point of view, though less elegant and compact than its predecessors. The construction is evident from the figure. M M are the magnets, now of the horse-shoe shape, A and B D the polepieces. This form of circuit has been adopted mainly in order to increase the induction. The previous forms do not lend themselves to the use of large cross-sections of steel without becoming inconvenient in shape. In this pattern it is evident that any desired quantity of steel may be used.

On the soft-iron cylinder B is mounted the brass frame carrying the entire moving system. The air-gap is adjusted

* See Proc. Phys. Soc. vol. xi. p. 306.

for uniformity of induction by means of the D portion of the polepiece, the details of which for the sake of simplicity are not given in the figure.

Fig. 3.



The latest form of Magnetic Circuit.

Considering next the electric circuit. The main parts of this are the moving coil and the spiral springs. The coil is wound on an aluminium rectangle, which serves the double purpose of a frame to stiffen the coil and a circuit of extremely low resistance for damping.

In the voltmeter the moving coil contains some 100 or 200 turns of the finest wire, while the amperemeter coil has some 20 turns of a moderately thick wire.

In series with the voltmeter moving coil there is placed in the base of the instrument a resistance-coil, on the value of which depend the readings of the meter.

In parallel with the amperemeter-coil is placed a short length of conductor of a similar material to that of the moving coil. Through this shunt flows the main current in all meters except the centiampere and milliampere ones.

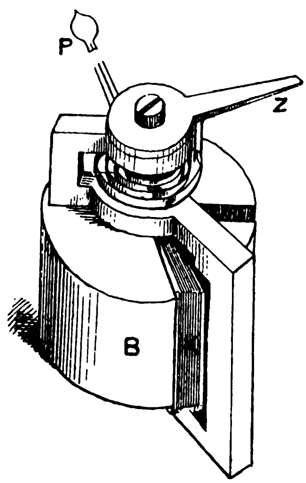
The current is led into and out of the moving coil by the spiral springs, which also provide the opposing or controlling force. To partially counterpoise the coil it is placed—with

respect to the axis—opposite the pointer. The coil and pointer are shown in position in the plan given in fig. 3.

A coil of this kind, swung on one of its sides, was used long ago by Lord Kelvin in his Siphon Recorder. It is also used at the present time by Professor Ayrton in the now well-known Astatic Station Voltmeter.

Fig. 4 gives a better view of coil, springs, and pointer, and the general method of mounting the entire moving system on the iron cylinder. K is the coil, P the pointer, and Z the

Fig. 4.



The central polepiece to which is attached the brass frame supporting the coil, pointer, and spiral springs.

arm for zero adjustment. There is here a compactness that becomes important in practice, for by loosening or tightening one screw the moving system with its frame can be removed bodily and placed in position again with ease, and without interfering with the magnetic circuit. The method also renders the balancing of the coil an easy matter.

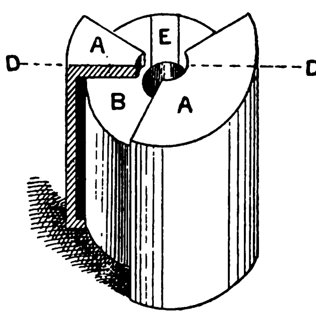
In all the meters, when the air-gap is made uniform in thickness, the induction density in the gap has practically a constant value throughout its entire length (except in the immediate neighbourhood of the two ends owing to the effect of prominences). This is the case even for the magnetic

circuits of figs. 1 and 2, where the induction is rather low, viz. (about) 400 per sq. cm. in the gap. With a magnetic flux of 400 the reluctance of the air is about 300 times that of the iron; so that the lines distribute themselves pretty uniformly on the polar surfaces if the intervening gap is made uniform in thickness. For the higher inductions of 800 or 1000 (for which the magnetic circuit of fig. 3 is designed) the magnetic force in the gap becomes still more uniform, for now we have the reluctance of the air 500 times that of iron.

There is another arrangement of polepieces which I would like to describe, although it has not yet been made into an instrument. It furnishes a great deflecting moment, but it has the rather serious disadvantage of being complex in design and—I am afraid—difficult to build. This arrangement, like the previous ones, may be attached either to a bar or a horse-shoe magnet, and may be arranged to give with a single moving coil a maximum deflexion of 220°, or it may be designed to give 80° or 90° with a double or twin coil. It may also be designed for small deflexions with a spot of light to be used as a galvanometer, in which case there is no difficulty at all in the construction.

Fig. 5 shows the form of the 220° polepieces without the magnet. The coil is entirely enclosed in iron, and its three

Fig. 5.



sides are active. In all other respects the arrangement is precisely the same as the previous ones.

Let fig. 6 represent a vertical section through D D′ in fig. 5, and let fig. 7 be the movable coil.

Fig. 6.

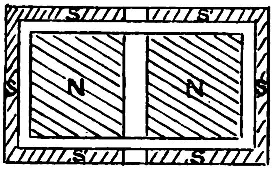
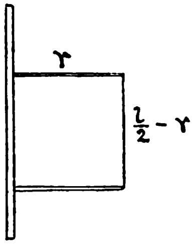


Fig. 7.



The lines of force cut the coil perpendicularly at all points. Let l be the length of the periphery of the coil, r the radial side, and n the number of turns in it. Considering one of the horizontal sides of the coil with a current c flowing in it, the moment of force acting at a distance x on an element dx is

$$=\mathrm{H}n\, cx\, dx.$$

The moment acting on the vertical side is

$$=\mathrm{H}nc\left(\frac{l}{2}-r\right)r,$$

H being the intensity of the magnetic field.

Therefore, the total deflecting moment M acting on the coil is

$$=\mathrm{H}\,nc\left[\left(\frac{l}{2}-r\right)r+2\int_0^r x\,dx\right].$$

So $\mathrm{M}=\frac{1}{2}nc\mathrm{H}lr$.

Unlike the old coil (that considered in the first part of this paper) this new coil has a deflecting moment increasing always in proportion to its radius if the magnetic field is kept constant. The moment is therefore a maximum when r is as nearly $\frac{1}{2}l$ as the constructional details of the instrument will permit. For the old coil the maximum moment is got when $r=\frac{1}{4}l$, *i. e.*, when the coil is a square. I offer the description of this instrument partly on account of its leading to (what is to me) a novel form of a ballistic galvanometer.

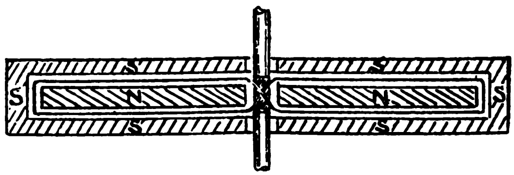
There may be more serious difficulties in the way of constructing a galvanometer on this plan than I at present anticipate.

Ballistic Galvanometer.

We evidently have in the arrangement shown in figs. 5, 6, and 7 favourable conditions for a ballistic galvanometer, since the deflecting moment and the moment of inertia of the coil attain their maximum values for the same value of the radius. The period of swing and the deflecting moment may therefore be increased simultaneously without making any addition to the weight of the moving system.

Let fig. 8 be a section through the polepieces of a ballistic

Fig. 8.



arrangement on the above plan, the coil completely embracing the central polepiece NN, and the outer polepiece S completely surrounding both. Here, of course, there is to be no aluminium frame for damping.

To compare the sensibility of this coil with another coil of the same mass, *effective area*, and number of turns, wound and suspended in the usual mode, let fig. 9 represent one, and

Fig. 9.

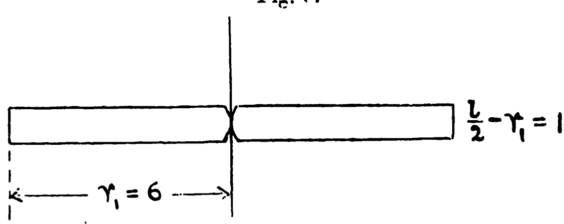
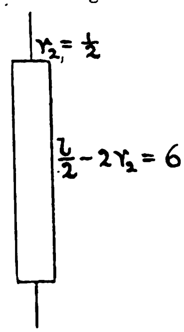


fig. 10 the other coil. Let $r_1 = 6$ in the first case, and let the depth of the coil be 1. In the second, let $r_2 = \frac{1}{2}$ and the

depth of the coil $=6$. Considered alone, the two coils are therefore similar in all respects. They are moreover supposed to be suspended in two fields of equal intensity, and to convey the same current.

Fig. 10.



For the sake of symmetry the wire in figs. 8 and 9 has to be divided between two rectangles, which, however, does not affect the argument.

The suspension in both cases having to bear the same load, the relative sensibilities of the two arrangements for continuous currents will therefore be as their deflecting-moments directly.

For fig. 9 the moment

$$M_1 = \tfrac{1}{2} ncHlr_1,$$

and for fig. 10

$$M_2 = 2ncH\left(\frac{l}{2} - 2r_2\right)r_2,$$

so that

$$\frac{M_1}{M_2} = \frac{lr_1}{4\left(\frac{l}{2} - 2r_2\right)r_2} = 7\,;$$

therefore $M_1 = 7M_2$ for steady currents.

Time of swing of the two coils.—The time of the swing varies with the $\sqrt{\text{moment of inertia}}$.

In fig. 9 the moment of inertia is

$$I_1 = mr_1^2\left(\frac{l}{2} - \frac{r_1}{3}\right),$$

and for fig. 10

$$I_2 = 2mr_2^2\left(\frac{l}{2} - \frac{4r_2}{3}\right)$$

where m stands for the mass of unit length of the periphery.

Therefore

$$\frac{I_1}{I_2} = \frac{r_1^2(3l - 2r_1)}{2r_2^2(3l - 8r_2)}.$$

Hence for the particular size of coils under consideration,

$$\frac{I_1}{I_2} = \frac{1080}{19}.$$

So that $$I_1 = 57\, I_2 \text{ nearly}.$$

Therefore, the time of swing of the coil in fig. 9 will be 7·5 times that for the coil in fig. 10, and the deflecting-moment or the sensitiveness of the coil in fig. 9 for continuous currents will be 7 times that of the coil in fig. 10.

Now as regards the "throw." The relative throws of the two coils for the same quantity of electricity will be

$$= \frac{M_1}{M_2} \cdot \sqrt{\frac{I_2}{I_1}} = \frac{7}{7{\cdot}5} \text{ about.}$$

So that we have for the two coils throws of nearly equal magnitudes, although one coil moves 7·5 times as slowly as the other.

It is almost needless for me to mention that I have been greatly assisted during the experimental period of the measuring instruments described above, and I wish to express my gratitude to Professor Oliver Lodge not only for his suggestions and criticisms but also for his kind encouragement at all times. I am also greatly indebted to Dr. Alexander Muirhead for much help in practical details, and for the considerable experimental facilities he has always been ready to present.

Discussion.

Prof. Ayrton said the instruments appeared to be very successful; he could bear witness of their value particularly as regards the length of range. The general principle by which long range was to be obtained on moving-coil, portable instruments was developed some ten years ago by

M. Carpentier, of Paris, who used a central magnet surrounded by a concentric hollow cylinder, with only one side of the coil in the magnetic gap between them; but it was not then a portable form of instrument, for the coil was suspended. Prof. Ayrton and Mr. Mather had worked in this direction in their "astatic station voltmeter." In that instrument there were three magnetic circuits arranged to give astaticism; this was described in 1892 or 1893.

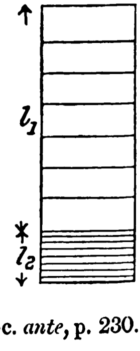
XLII. *Note on the Source of Energy in Diffusive Convection.* By ALBERT GRIFFITHS, *M.Sc.* (*Vic.*), *A.R.C.S.* (*Lond.*).*

AT the conclusion of a paper on "Diffusive Convection"† the author, partly in the hope of producing a discussion, asked certain questions relating indirectly to the source of energy in the apparatus under consideration.

After the publication of the paper in the Philosophical Magazine, Prof. FitzGerald made some remarks on it in 'Nature,' and gave a concise account of the actions at work. He pointed out, what was already known to the author, that there is a tendency towards cooling when diffusion causes the rise of the centre of gravity. Stimulated by Prof. FitzGerald's interest in the work, the author has taken advantage of the Christmas vacation to study the question in some detail.

On the Fall of Temperature when Diffusion occurs upwards if the Solution has a Density greater than that of Water.

Fig. 1.



Consider a vertical cylinder of length (l_1+l_2) and sectional area A. Let d_2 equal the density of length l_2, and d_1 the density of length l_1. Let it be assumed that there is no change of volume when the two liquids mix.

Taking a plane through the bottom of the cylinder as one where bodies at rest possess zero potential energy, it can readily be seen that before mixing, through diffusion or otherwise,

* Read March 10, 1899.

† Phil. Mag. ser. 5. vol. xlvi. p. 453 (1898); Proc. Phys. Soc. *ante*, p. 230.

the potential energy

$$=d_2 A g \frac{l_2^2}{2} + d_1 A g l_1 \left(l_2 + \frac{l_1}{2}\right).$$

The liquids also possess energy due to the heat they contain.

Let S_1 = thermal capacity of unit mass of liquid of density d_1.
S_2 = " " " solution " d_2.
S = " " " mixture.
t_0 = temperature (absolute scale) before mixing.
t_1 = " " after "

The *energy*, before mixing, *in the form of heat* equals

$$\int_0^{t_0} d_1 A l_1 S_1 dt + \int_0^{t_0} d_2 A l_2 S_2 dt.$$

The combined potential and thermal energy equals

$$d_2 A g \frac{l_2^2}{2} + d_1 A g l_1 \left(l_2 + \frac{l_1}{2}\right) + \int_0^{t_0} d_1 A l_1 S_1 dt + \int_0^{t_0} d_2 A l_2 S_2 dt.$$

Let mixing take place without the addition or withdrawal of heat. Neglecting the heat of combination, the expression for the combined energy now equals

$$\frac{g(d_1 A l_1 + d_2 A l_2)(l_1 + l_2)}{2} + \int_0^{t_1} (d_1 A l_1 + d_2 A l_2) S dt.$$

Equating the two expressions, dividing throughout by A, and rearranging,

$$\frac{g l_1 l_2 (d_1 - d_2)}{2} + \int_0^{t_0} d_1 l_1 S_1 dt + \int_0^{t_0} d_2 l_2 S_2 dt = \int_0^{t_1} (d_1 l_1 + d_2 l_2) S dt.$$

If we make the hypothesis that S_1, S_2, and S are constants, then

$$\frac{g l_1 l_2 (d_1 - d_2)}{2} = S(d_1 l_1 + d_2 l_2) t_1 - d_1 l_1 S_1 t_0 - d_2 l_2 S_2 t_0.$$

This equation indicates that $(t_0 - t_1)$, the fall in temperature, depends in some way on the height through which the

dissolved substance diffuses. The preceding was worked out long before I published my paper. In what follows, the influence of Prof. FitzGerald will perhaps be shown.

If we wish to study the fall of temperature due to the rise of matter by diffusion in a theoretical manner, we may assume that the capacity for heat of the mixture is the sum of the capacities for heat of the components, and that there is no heat of combination.

Let $m =$ total mass of the two liquids.
s = thermal capacity of unit mass of mixture.
h = rise of C.G.
g = acceleration due to gravity.
f = the fall of temperature.

Then, since the potential energy gained equals the capacity of the liquids multiplied by the fall of temperature,

$$mgh = msf.$$

Hence
$$f = \frac{gh}{s}.$$

It may be pointed out that this equation indicates a limit to the height to which diffusion (with ultimate uniform distribution) can spread. Thus if s be a constant, the above equation holds whatever the fall in temperature; and if the temperature be initially 0° C., f cannot be greater than 273° C., and h cannot be greater than 273 s/g.

Let $s = 42{,}350 \times g$ (the capacity of a gram of water), then h cannot be greater than $273 \times 42{,}350$ cms.; that is 115 kilometres approximately. It may be noted that h is not the height at which diffusion occurs, but the rise in the centre of gravity. In a given case the height at which diffusion occurs can readily be expressed in terms of h.

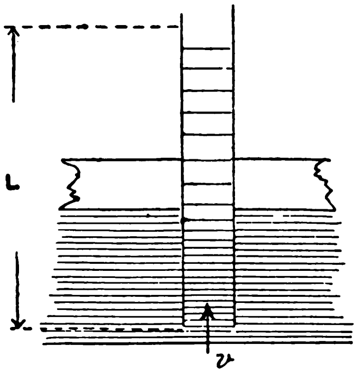
The Absorption of Heat in a simple case when there is a combination of Diffusion and Motion.

In the figure there is a representation of a vertical tube of length L.

Let us suppose that the lower end is continuously in contact

with an aqueous solution of concentration T, and that its upper extremity is in contact with pure water.

Fig. 2.



Let v = velocity of liquid up the tube.
A = sectional area of the tube.

With certain assumptions, it is shown in the paper on "Diffusive Convection" that the difference of pressure between the bottom and the top of the tube equals

$$gL + \frac{gTL}{1-e^{-\frac{vL}{k}}} - \frac{kgT}{v}.$$

The external work done on the cylinder per second equals the product of the volume which crosses any section and the difference of pressure; *i. e.*, it equals

$$vA\left(gL + \frac{gTL}{1-e^{-\frac{vL}{k}}} + \frac{kgT}{v}\right)$$

$$= vAgL + \frac{vAgTL}{1-e^{-\frac{vL}{k}}} + kgTA.$$

The work performed per second consists of two parts—

(1) A quantity of water of volume vA is raised a height L.

(2) A quantity of dissolved substance (see "Diffusive Convection," Section II.) of amount $\dfrac{vAT}{1-e^{-\frac{vL}{k}}}$ is raised a height L.

Let H = the heat absorbed in unit time; we have:—
External work done − H = work performed: hence

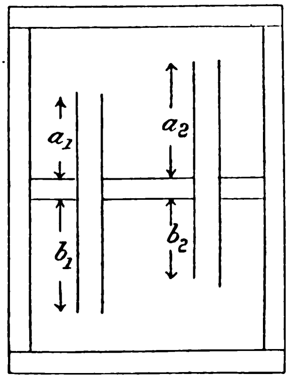
$$vAgL + \frac{vAgTL}{1-e^{-\frac{vL}{k}}} + kgTA - H = vAgL + \frac{vATgL}{1-e^{-\frac{vL}{k}}},$$

and $$H = kgTA.$$

Thus the heat absorbed through the agency of the tube is independent of the height of the tube and of the velocity of flow.

Calculation of the Rate at which Work is done by the Apparatus described in "Diffusive Convection."

Fig. 3.



The apparatus to be studied is illustrated in fig. 3.

Let L_1 = length of left-hand tube.
L_2 = " right "
A = area of each tube.
c = thickness of the diaphragm.
a_1 = difference of level between the top of L_1 and the top of the diaphragm.
a_2 = difference of level between the top of L_2 and the top of the diaphragm.

b_1 = distance between the bottom of L_1 and the bottom of the diaphragm.

b_2 = distance between the bottom of L_2 and the bottom of the diaphragm.

The work done by the apparatus is equal to the volume which crosses a section in unit time multiplied by what in Section II. of "Diffusive Convection" is called "the pressures tending to produce circulation."

The work done per second equals

$$vA\left[\left\{gL_2+\frac{gTL_2}{1-e^{\frac{vL_2}{k}}}+\frac{kgT}{v}+(b_1-b_2)(1+T)g\right\}\right.$$
$$\left.-\left\{(a_2-a_1)g+gL_1+\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}-\frac{kgT}{v}\right\}\right];$$

or noting that $L_1=a_1+c+b_1$, $L_2=a_2+c+b_2$, it equals

$$vA\left\{\frac{2kgT}{v}+\frac{gTL_2}{1-e^{\frac{vL_2}{k}}}-\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}+g(b_1-b_2)T\right\}.$$

It may be mentioned that the work done equals zero when $v=0$ (in this case there is an impervious obstruction to the flow); and when

$$\frac{2kgT}{v}+\frac{gTL_2}{1-e^{\frac{vL_2}{k}}}-\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}+g(b_1-b_2)T=0$$

(if $\frac{vL}{k}$ is small), *i. e.* when

$$v=\frac{6k(\overline{a_2-a_1}+\overline{b_1-b_2})}{L_1^2+L_2^2}.$$

(In this case there is no obstruction whatever to the flow.)

The work done by the apparatus is a maximum when v is approximately half the latter value. Work is done on the apparatus when v is negative, or v is greater than

$$6k(\overline{a_2-a_1}+\overline{b_1-b_2})/(L_1^2+L_2^2).$$

Calculation, by means of Energy Equations, of the Rate at which Work is done.

Before tackling the problem, it will be well perhaps to make some preliminary remarks with regard to what occurs in the upper and lower compartments. The dissolved substance is being continually carried to the top of L_1, for example, and falling from there to the bottom of the upper compartment. The heat generated by the fall equals the product of the weight transmitted into the fall.

On the other hand, when the dissolved substance is abstracted from the bottom, the weakened solution rises to the top of the lower compartment. Diffusion into the weakened solution, whilst it is rising, will doubtless occur, *but the action can be made negligible by taking* g *very large*, in which case the weakened solution rises very rapidly.

If $d=1+t$ (where d=density and t=concentration), it can readily be shown that the heat generated equals the *weight transmitted* into the bottom of the tube *by diffusion* multiplied by the rise of the weakened solution.

The apparatus sketched in the preceding section will now be studied.

The quantity transmitted through L_1 into the upper compartment equals

$$\frac{vAT}{1-e^{-\frac{vL_1}{k}}}.$$

The heat produced by the fall of the substance transmitted to the bottom of the upper compartment equals

$$\frac{vATga_1}{1-e^{-\frac{vL_1}{k}}}.$$

The quantity transmitted into the bottom of L_1 *by diffusion* equals

$$\frac{vAT}{1-e^{-\frac{vL_1}{k}}}-vAT=\frac{vATe^{-\frac{vL_1}{k}}}{1-e^{-\frac{vL_1}{k}}}.$$

The energy produced by the rise of the diluted solution equals

$$\left(\frac{v\mathrm{AT}ge^{-\frac{v\mathrm{L}_1}{k}}}{1-e^{-\frac{v\mathrm{L}_1}{k}}}\right)b_1.$$

Proceeding in this way, and making some algebraic transformations, it can readily be shown that the heat produced in this way when we consider the two tubes equals

$$v\mathrm{AT}\left\{\frac{a_1+b_1e^{-\frac{v\mathrm{L}_1}{k}}}{1-e^{-\frac{v\mathrm{L}_1}{k}}}-\frac{a_2+b_2e^{\frac{v\mathrm{L}_2}{k}}}{1-e^{\frac{v\mathrm{L}_2}{k}}}\right\}.$$

One effect of the operations just considered is to reduce the quantity of the dissolved substance at the top of the lower compartment by a certain amount, and to place the same amount at the bottom of the upper compartment.

The potential energy gained per second is the product of the weight of the total amount transmitted and the thickness of the diaphragm. Thus the potential energy gained per second equals

$$g\left(\frac{v\mathrm{AT}}{1-e^{-\frac{v\mathrm{L}_1}{k}}}-\frac{v\mathrm{AT}}{1-e^{\frac{v\mathrm{L}_2}{k}}}\right)c.$$

Let W = rate at which work is done *by* the apparatus.

D = rate at which heat is absorbed due to diffusion along the tubes.

F = rate at which heat is produced by the fall of the dissolved substance in the upper compartment, and the rise of weakened solution in the lower.

V = rate at which potential energy is gained by the carriage of dissolved substance from the bottom to the top of the diaphragm.

Then

$$\begin{aligned}\mathrm{W}&=\mathrm{D}-\mathrm{F}-\mathrm{V}\\&=2k\mathrm{AT}g-v\mathrm{AT}\left\{\frac{a_1+b_1e^{-\frac{v\mathrm{L}_1}{k}}}{1-e^{-\frac{v\mathrm{L}_1}{k}}}-\frac{a_2+b_2e^{\frac{v\mathrm{L}_2}{k}}}{1-e^{\frac{v\mathrm{L}_2}{k}}}\right\}\\&\qquad-\left(\frac{v\mathrm{AT}}{1-e^{-\frac{v\mathrm{L}_1}{k}}}-\frac{v\mathrm{AT}}{1-e^{\frac{v\mathrm{L}_2}{k}}}\right)c.\end{aligned}$$

Noting that $a_1+c=L_1-b_1$, $a_2+c=L_2-b_2$, it can be shown algebraically that

$$W=vA\left\{\frac{2kgT}{v}+\frac{gTL_2}{1-e^{\frac{vL_2}{k}}}-\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}+g(b_1-b_2)T\right\}.$$

This is exactly the same result as that obtained in an earlier section.

General Explanation of the Loss of Heat when Work is done by the Apparatus.

For simplicity let the thickness of the diaphragm be zero.

Transmission of dissolved substance along the tubes is accompanied by a fall of substance in the upper compartment, and a rise of weakened solution in the lower compartment, both of which actions produce heat. The rate of production of heat depends on the rate of flow of the liquid.

The *greater* the rate at which work is done by the apparatus, the less the rate of production of heat.

The rate of absorption of heat due to transmission of dissolved substance through the tubes is independent of the flow, and need not be considered in this general explanation.

XLIII. *A Study of an Apparatus for the Determination of the Rate of Diffusion of Solids dissolved in Liquids.* By ALBERT GRIFFITHS, *M.Sc.(Vic.), A.R.C.S.(Lond.)*.*

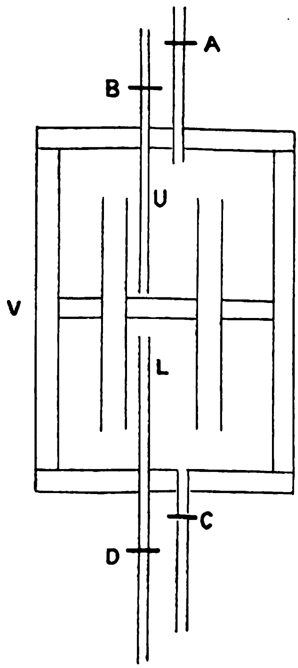
SECTION I.

FOR some time the author has been engaged in some experiments with the object of determining the coefficient of diffusion of bodies dissolved in water. It may be some years before experimental results of any decided value are obtained; and perhaps he will be allowed to give an account of the calculations involved, and of the methods he has adopted in obtaining an idea of the probable order of magnitude of the errors to which the apparatus is liable.

* Read March 10, 1899.

The apparatus consists of a vessel V, divided into two compartments, U and L, by a diaphragm through which pass a number of equal vertical tubes of which two only are shown in the figure. Two tubes, A, B, provided with stop-cocks, pass into U. One, A, just enters U; the other passes down to the bottom. Similarly two tubes C and D pass into L; C just enters, and D reaches to the top. The vessel is first completely filled with water, and in the case of a substance which produces an aqueous solution with a specific gravity greater than unity, the tubes A and B are closed, and the solution is passed in through C, the water of L being allowed to escape through D. Diffusion along the vertical tubes now commences, and the compartments U and L are alternately and periodically refilled with pure water and solution respectively. A quantitative analysis is made of the liquid taken from the upper compartment.

Fig. 1.



SECTION II.—*Calculation of the Coefficient of Diffusion.*

Lord Kelvin has solved the problem of the flow of electricity along a cable possessing appreciable capacity. This solution can readily be transferred to the problem under consideration.

It will be assumed that the upper extremities of the tubes are kept in contact with pure water, and the lower extremities in contact with a solution of constant strength.

Let γ = quantity of dissolved substance which enters the upper compartment in unit time, when the

combined *sectional area* of the tubes equals *unity*.

k = coefficient of diffusion (C.G.S. system), assumed to be a constant.

L = length of tubes in centims.

T = quantity of substance per c. c. in the lower compartment.

t = time in seconds.

Then

$$\gamma = \frac{k\mathrm{T}}{\mathrm{L}}\left\{1 + 2\sum_{i=1}^{i=\infty}(-1)^i e^{\frac{-i^2\pi^2 kt}{\mathrm{L}^2}}\right\},$$

where i is any integer.

[If $\frac{\mathrm{L}^2}{\pi^2 k}\log\frac{4}{3}$ be taken as the unit of time, the topmost of the series of curves given on page 72, vol. ii., of Kelvin's 'Mathematical and Physical Papers,' shows graphically how γ rises to a maximum.]

Let q = total quantity of substance transmitted in t seconds. Integrating the preceding expression with respect to t, we obtain

$$q = \frac{k\mathrm{T}}{\mathrm{L}}\left\{t + 2\sum_{i=1}^{i=\infty}(-1)^i\frac{\mathrm{L}^2}{i^2\pi^2 k} - 2\sum_{i=1}^{i=\infty}(-1)^i\frac{\mathrm{L}^2}{i^2\pi^2 k}e^{\frac{-i^2\pi^2 kt}{\mathrm{L}^2}}\right\} \quad (1)$$

When t *is very large* the third term within the brackets may be neglected, and

$$q = \frac{k\mathrm{T}t}{\mathrm{L}} + \frac{2\mathrm{LT}}{\pi^2}\sum_{i=1}^{i=\infty}\frac{(-1)^i}{i^2} \quad . \quad . \quad . \quad . \quad (2)$$

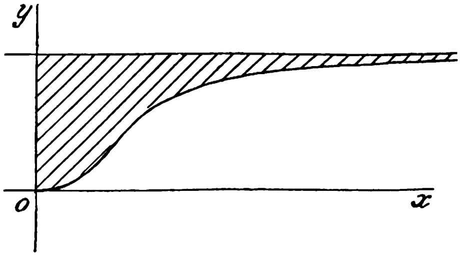
or

$$q = \frac{k\mathrm{T}t}{\mathrm{L}} - \frac{2\mathrm{LT}}{\pi^2}\frac{\pi^2}{12} = \frac{k\mathrm{T}t}{\mathrm{L}} - \frac{\mathrm{LT}}{6}. \quad . \quad . \quad . \quad (3)$$

The last equation shows that if t is large the quantity transmitted equals $k\mathrm{T}t/\mathrm{L}$ minus a quantity which is independent of the coefficient of diffusion. Thus if in the adjoining figure abscissæ represent times, and ordinates the quantities

diffused per second, the shaded area between the asymptote and the curve is independent of the coefficient of diffusion.

Fig. 2.



The table below gives an idea of the rate at which the steady state is attained when $L=4{\cdot}043$, and $k=3\times10^{-6}$. The unit of quantity is the amount transmitted per week on the attainment of the steady state.

No. of Weeks.	Total Quantity.	Quantity per Week.
1	0·104	0·104
2	0·703	0·599
3	1·567	0·864
4	2·519	0·952
5	3·505	0·986
6	4·500	0·995
7	5·498	
8	6·498	
9	7·498	
10	8·498	

[The above table gives the *integral* quantities transmitted. The curve already mentioned shows the way in which the rate of transmission varies. The unit abscissa $\frac{L^2}{\pi^2 k}\log_e\frac{4}{3}=$ ·2626 of a week.]

SECTION III.—*Error due to Differences of Temperature.*

The object of this section is to obtain the *order of magnitude* of the errors produced by differences in temperature between the various tubes.

For simplicity, two tubes only will be considered whose lengths and sectional areas are the same. The expansion of the material of the tubes will be neglected. To make the problem amenable to mathematical treatment, it will be assumed that the density of water equals $(1-\alpha\theta)$, where θ is the temperature, and α is approximately equal to an ideal coefficient of expansion. It will also be assumed that

$$d=(1-\alpha\theta+t),$$

where $d=$ the density of the solution,

$t=$ the concentration.

Let $\theta_1=$ the temperature of one tube.
$v_1=$ the velocity of the liquid *up* the tube.
$\theta_2=$ the temperature of the second tube.
$v_2=$ the velocity of the liquid *up* the tube.
$\mathrm{L}=$ the length of each tube, assumed constant.
$k=$ the coefficent of diffusion, assumed constant.
$\mathrm{T}=$ the concentration of the solution at the bottom of each tube.

From considerations similar to those adopted in "Diffusive Convection"*, Sect. II. and III., it can be shown that, neglecting viscosity,

$$g(1-\alpha\theta_1)\mathrm{L}+\frac{g\mathrm{TL}}{1-e^{-\frac{v_1\mathrm{L}}{k}}}-\frac{kg\mathrm{T}}{v_1}$$
$$=g(1-\alpha\theta_2)\mathrm{L}+\frac{g\mathrm{TL}}{1-e^{-\frac{v_2\mathrm{L}}{k}}}-\frac{kg\mathrm{T}}{v_2}.$$

Dividing throughout by g, and expanding,

$$(1-\alpha\theta_1)\mathrm{L}+\mathrm{TL}\left(\frac{k}{v_1\mathrm{L}}+\frac{1}{2}+\frac{v_1\mathrm{L}}{12k}+\&c.\right)-\frac{k\mathrm{T}}{v_1}$$
$$=(1-\alpha\theta_2)\mathrm{L}+\mathrm{TL}\left(\frac{k}{v_2\mathrm{L}}+\frac{1}{2}+\frac{1}{12}\frac{v_2\mathrm{L}}{12k}+\&c.\right)-\frac{k\mathrm{T}}{v_2};$$

* *Ante*, pp. 230–243; Phil. Mag. Nov. 1898, pp. 453–465.

and

$$(v_2 - v_1) = \frac{12k\alpha(\theta_2 - \theta_1)}{\mathrm{TL}}, \text{ approx.}$$

Now

$$\frac{v_1}{v_2} = -\left(\frac{1-\alpha\theta_2}{1-\alpha\theta_1}\right);$$

hence

$$v_2\left\{1 + \frac{1-\alpha\theta_2}{1-\alpha\theta_1}\right\} = \frac{12k\alpha(\theta_2 - \theta_1)}{\mathrm{TL}},$$

and

$$\frac{v_2\mathrm{L}}{k} = \frac{12\alpha(1-\alpha\theta_1)(\theta_2-\theta_1)}{\mathrm{T}(2-\alpha\overline{\theta_1+\theta_2})};$$

similarly

$$\frac{v_1\mathrm{L}}{k} = \frac{12\alpha(1-\alpha\theta_2)(\theta_1-\theta_2)}{\mathrm{T}(2-\alpha\overline{\theta_1+\theta_2})}.$$

The correcting factor equals

$$\frac{\left\{1+\frac{1}{2}\frac{v_1\mathrm{L}}{k}+\frac{1}{12}\frac{v_1\mathrm{L}^2}{k^2}\right\}+\left\{1+\frac{1}{2}\frac{v_2\mathrm{L}}{k}+\frac{1}{12}\frac{v_2{}^2\mathrm{L}^2}{k^2}\right\}}{2}$$

$$=1+\frac{1}{4}\left(\frac{v_1\mathrm{L}}{k}+\frac{v_2\mathrm{L}}{k}\right)+\frac{1}{24}\left(\frac{v_1{}^2\mathrm{L}^2}{k^2}+\frac{v_2{}^2\mathrm{L}^2}{k^2}\right)$$

$$=1+\frac{3}{2}\frac{\alpha^2(\theta_2-\theta_1)^2}{\mathrm{T}}+\frac{3\alpha^2(\theta_2-\theta_1)^2}{\mathrm{T}^2}, \text{ approx.}$$

As an example, if $(\theta_2-\theta_1)=0°\cdot1$ C., T=0·1, and $\alpha=0\cdot00015$, the correcting factor equals 1·00000007.

When T is a small fraction of unity, as in general will be the case, the correcting factor equals

$$1+\frac{3\alpha^2(\theta_2-\theta_1)^2}{\mathrm{T}^2}, \text{ approx.}$$

It may be noted that, approximately,

$$v_2 = -v_1 = \frac{12k\alpha(\theta_2-\theta_2)}{\mathrm{TL}}.$$

As an example, let $k=3\times10^{-6}$, $\alpha=0\cdot00015$, $(\theta_2-\theta_1)=0\cdot1$, T=0·1, L=4 cm.

Then

$$v_2 = 1{\cdot}35 \times 10^{-9} \text{ cm. per second,}$$
$$= 0{\cdot}04 \text{ cm. per year.}$$

SECTION IV.—*Errors due to Changes of Temperature of Apparatus as a whole.*

Apart from the effects of any differences of temperature which may exist between the tubes themselves, there will in general be a flow set up along them one way or the other when the apparatus as a whole is heated.

If the upper and lower compartments were of equal volume and the coefficients of expansion and the compressibilities of water and solution were the same, there would be no movement along the tubes; the liquid would simply be compressed slightly. The actual motion could only be calculated from a knowledge of the various coefficients of expansion and the compressibilities. The author has not the data for an accurate calculation, but to give an idea of the order of magnitude of the error, the case will be considered in which the taps of the upper compartment are open.

Let V = volume of lower compartment.
α = coefficient of cubical expansion of liquid.
θ = rise in temperature per second.

It will be assumed that the temperature increases at a constant rate, and that the steady state has been attained.

Let v = velocity *up* the tubes per second.
L = length of each tube.

Then $v = \dfrac{V\alpha\theta}{A}$.

The quantity of dissolved substance transmitted per second (see Diffusive Convection, Sec. IV.) equals, when VL/k is small,

$$\frac{kAT}{L}\left(1 + \tfrac{1}{2}\frac{vL}{k}\right)\text{approx.,}$$

$$\textit{i. e.}\quad \frac{kAT}{L}\left(1 + \tfrac{1}{2}\frac{VL\alpha\theta}{Ak}\right).$$

As an example, let the rise of temperature be 2° C. per week; let $\alpha=0{\cdot}00015$, $L=4$ cm., $V=40$ c.c., $A=0{\cdot}4$, and $k=3\times10^{-6}$. Then $VL\alpha\theta/2Ak$ equals 0·033; and the error due to the flow produced by the expansion is more than 3 per cent.

It may be mentioned that if a bubble of air is present in the lower compartment, it increases the flow, and the corresponding error, considerably.

Even if the upper taps are not left continuously open, there is an error corresponding to the above; for opening the top taps will allow a sudden rush to take place up the tubes.

It is obvious that a diminution of temperature produces an error in the opposite direction; and the error produced by a fluctuating temperature will probably not be considerable in the long run.

SECTION V.—*Errors due to Changes in Volume produced by the Weakening of the Solution through Diffusion.*

In general, when a solution loses some of the dissolved substance by diffusion it diminishes in volume. The diminution depends on the nature of the dissolved substance and varies with the concentration. The diminution for any given loss can be calculated from a table of densities. For example, an aqueous solution of copper sulphate at a temperature of 23°·3 containing 80 grams to the litre has a density of 1·080, whereas a solution containing 160 grams to the litre has a density of 1·154. It can be readily shown that the addition of 1 gram of anhydrous copper sulphate, between the given limits, on the average increases the volume by 0·075 c.c. approx.

As in Section IV. it will be assumed that the upper compartment is open to the atmosphere.

Since the diminution of volume of the lower compartment equals the quantity of copper sulphate transmitted multiplied by 0·075, with the aid of Sec. I. "Diffusive Convection," it can readily be seen that

$$v=\frac{-0{\cdot}075\times T\times v}{1-e^{-\frac{vL}{k}}}.$$

Hence, neglecting the second and higher powers of $\frac{vL}{k}$,

$$\frac{vL}{k}=0{\cdot}075\times T.$$

Let $\qquad T=0{\cdot}12, \quad \text{then } \frac{vL}{k}=0{\cdot}009.$

The fractional diminution in the quantity transmitted equals $vL/2k$, i.e. 0·0045. Thus the motion along the tubes causes a diminution in the quantity transmitted by about one half per cent.

With salts whose solutions experience greater changes in volume, the error may be very appreciable.

Thus, between the limits of density 1·081 and 1·159, the addition of 1 gram of sodium chloride causes an average increase in volume of 0·35 c.c.; and with $T=0{\cdot}12$, the error would be 2·1 per cent.

It may be noted that when a small quantity of copper sulphate is added to water, the resultant volume is about the same as the original volume of the water. Hence if the lower compartment were left open to the atmosphere instead of the upper one, there would be no error from the cause under consideration.

As before errors are produced when the taps are opened intermittently.

SECTION VI.—*Errors due to Circulations produced when the Liquids of the Upper and Lower Compartments are renewed.*

When the water in the upper compartment is renewed the pressure at the inlet is greater than the pressures at the outlet, and in consequence the pressures at the upper extremities of the tubes are not constant. Hence flows take place up certain tubes and down others.

Let there be n tubes; and let p_1, p_2, &c. be the respective *excesses* of the pressures at the tops of the tubes above the *average* pressure. Assuming the attainment of the steady state, and making the usual approximations, it can readily be shown that the correcting factor equals

$$1+\frac{12}{g^2T^2L^2}\left(\frac{p_1{}^2+\ldots p_n{}^2}{n}\right).$$

Let p_r be the greatest of the p's ; then the fractional error is less than $12\, p_r^2/g^2T^2L^2$. Neglecting inertia effects, the value of p_r must be less than the difference of pressure between the inlet and outlet, an estimate of which could be made practically. The author has not made any practical measurements, but has made a guess as to the order of magnitude of the difference of pressure by making a crude analogy between the upper compartment and a cylindrical tube. Thus, if 100 c.c. of water pass along a tube 3 cm. in diameter in an hour, the difference of pressure (assuming the formula for a capillary to hold) between two points on the axis 3 centim. apart, if we take the coefficient of viscosity equal to 0·01, is 0·004 dyne. Let $p_r = 0{\cdot}01$, $T = 0{\cdot}1$, $L = 4$, then the fractional error equals

$$12 \times (\cdot 01)^2/(981)^2(0\cdot 1)^2 \times 4^2, \text{ or } 7\cdot 8 \times 10^{-9}.$$

It is clear that the error from this cause is not appreciable.

SECTION VII.—*Error due to Diffusive Convection.*

Inequalities in the lengths of the tubes and inaccuracies in placing them in the diaphragm will, in general, produce convection-currents. There is no intrinsic difficulty in calculating flows of small magnitude, if the tubes are of unequal length ; but, to simplify the algebra, in the following the tubes will be taken of equal length and of equal sectional area.

Let n = number of tubes.

δ_1 = distance between the top of one tube and the *average* of the tops of all the tubes.

δ_r = corresponding distance for the rth tube.

v_1 = velocity up the first tube.

v_r = „ „ rth tube.

Making the usual assumptions, it can be shown that

$$\frac{(v_1 - v_r)L^2}{12k} = \delta_r - \delta_1.$$

Noting that $\Sigma v_r = 0$, $\Sigma \delta_r = 0$, it can readily be shown that

$$\frac{v_r L}{k} = \frac{-12\delta_r}{L},$$

and that the correcting factor equals

$$1 + \frac{12}{L^2} \frac{\delta_1^2 + \ldots \delta_n^2}{n}.$$

With ordinary care in the construction of the apparatus, the correcting factor will deviate very little from unity.

Section VIII.—*Some Experimental Results.*

One apparatus has 15 tubes, of average length 4·043 centim., and a total sectional area 0·4102 square centim. It was placed in a cellar the temperature of which fluctuated through a range of about 3° C. per week. The lower compartment contained a solution of copper sulphate holding 0·0506 of copper to the c.c. At intervals of a week the upper compartment was renewed with pure water; similarly the lower compartment was renewed with fresh solution. The tubes were originally full of pure water. The average temperature was about 8° C. By the end of the first week 0·0026 gram of copper had been transmitted; by the end of the second a *total* of 0·0077; end of third, 0·0144; end of fourth, 0·0226; end of fifth, 0·0311; end of sixth, 0·0403; end of eighth, 0·0559; end of ninth, 0·0662.

Taking as a basis the quantity transmitted in nine weeks, the coefficient of diffusion equals $28 \cdot 7 \times 10^{-7}$. This result, and others not recorded, indicate that the errors due to the various causes studied are, at any rate, not enormous.

To diminish the variations of temperature, in one experiment the whole apparatus was put inside a copper cylinder; but, unfortunately, some air-bubbles were accidentally allowed to form within the compartments, and the quantities of copper sulphate transmitted were irregular and of large value. At present the author is experimenting with the tubes full of a weak gelatine jelly, with the intention, ultimately, of stopping each tube with a short plug of jelly, and thus preventing circulations of all sorts. The plugs will probably be made insoluble with formaldehyde. Mr. Hibbert has helped me with the analyses, and the results are hopeful; but, as already indicated, progress is inevitably slow.

University College, Sheffield.

Discussion.

The President described an apparatus he had used for a thermostat. A double-walled cylinder of copper sheet, with a little water in the interspace, is exhausted at atmospheric temperature until the water boils; it is then sealed up. Water-vapour is a powerful equaliser of temperature, and a vapour-jacket of this kind is very efficient when it is required to maintain uniform temperature—not constant temperature.

Mr. Watson described an arrangement used by Mr. E. H. Griffiths in which tap-water is employed as a negative source and a gas-flame as a positive source—with extremely good results—as a thermostat for constant and for uniform temperatures of about 15° C.

XLIV. *The Joule-Thomson Thermal Effect; its Connexion with the Characteristic Equation, and some of its Thermodynamical Consequences.* By E. F. J. Love, *M.A., F.R.A.S., Assistant Lecturer and Demonstrator of Natural Philosophy in the University of Melbourne*.*

(1) *Introductory.*

The results obtained by Lord Kelvin and the late Dr. Joule in their famous investigation into the "Thermal Effects of Fluids in Motion" have hitherto been utilized almost exclusively for one special purpose, viz. for determining the relation between various gas-thermometer scales and the absolute scale of temperature. But this is not the only information that can be obtained from them; and the present paper has been written partly in order to deduce some of their further consequences.

This deduction, however, is not the writer's sole object. He desires to indicate the relation which must exist between the formula assigned to the Joule-Thomson effect, considered as a function of the temperature, and the particular form adopted for the characteristic equation of a gas: in this way it is

* Read February 24, 1899.

possible to supply some sort of theoretical basis to the various formulæ put forward by different investigators, and to obtain some idea as to which of them is the most likely to be correct. With a view to an orderly development of the subject this discussion will be taken first; it will be followed by a deduction, from Joule and Lord Kelvin's results, of the relation between the intrinsic energy of a gas and its volume, upon which is based a new method of calculating the ratio of the two principal specific heats of a gas. A concluding paragraph deals with some points in the thermodynamics of substances at their temperature of maximum density.

(2) *Notation.*

The following scheme of notation is adhered to in this paper:—

Pressure of a substance	p
Volume of unit mass	v
Absolute temperature	T
Density of substance	ρ
Specific heat at constant pressure	K_p
Specific heat at constant volume	K_v
Intrinsic energy of unit mass	U
Joule-Thomson cooling effect per atmosphere rise of pressure	θ
Standard pressure of 1 atmosphere	Π
Mechanical equivalent of heat	J
Coefficient of dilatation	σ
Quantity of heat	Q

The letters A, B, C, R, a, b, α, β, γ, δ denote constants occurring in various equations; subscribed letters indicate that the quantities denoted by them are supposed constant.

(3) *The Joule-Thomson Effect and the Gas-Equation.*

The reasoning of this section is based throughout on Lord Kelvin's well-known equation

$$T\left(\frac{\partial v}{\partial T}\right)_p - v = -J \, . \, K_p \frac{dT}{dp},$$

where dT denotes the rise of temperature in a "porous plug" experiment corresponding to a rise of pressure dp on passing the plug.

(1) If we adopt the ordinary Boyle-Charles equation of a perfect gas

$$pv = RT,$$

then we can show at once by differentiation that

$$T\left(\frac{\partial v}{\partial T}\right)_p - v = 0;$$

in other words, the Joule-Thomson thermal effect is zero, a result which the experiments conclusively disprove.

(2) A very different result is obtained if we adopt van der Waals's equation; as we shall see, it leads directly to the expression recently* formulated by Rose-Innes. We start from the equation

$$\left(p + \frac{a}{v^2}\right)(v-b) = RT;$$

dividing throughout by $v-b$, and differentiating with respect to T, on the assumption $p=$const., we obtain

$$-\frac{2a}{v^3}\left(\frac{\partial v}{\partial T}\right)_p = \frac{R}{v-b} - \frac{RT}{(v-b)^2}\cdot\left(\frac{\partial v}{\partial T}\right)_p;$$

$$\therefore \left(\frac{\partial v}{\partial T}\right)_p = \frac{\dfrac{R}{v-b}}{\dfrac{RT}{(v-b)^2} - \dfrac{2a}{v^3}},$$

i. e. $$T\left(\frac{\partial v}{\partial T}\right)_p = (v-b)\left\{1 - \frac{2a}{RTv}\left(1-\frac{b}{v}\right)^2\right\}^{-1}.$$

Multiplying out on the right-hand side and neglecting powers and products of a and b (since both are very small) we obtain

$$T\left(\frac{\partial v}{\partial T}\right)_p = v - b + \frac{2a}{RT}; \quad . \quad . \quad . \quad . \quad \text{(q. p.)}$$

whence

$$T\left(\frac{\partial v}{\partial T}\right)_p - v = \frac{2a}{RT} - b;$$

* Phil. Mag. March 1898.

i. e.
$$-\mathrm{J}\,.\,\mathrm{K}_p\,.\frac{d\mathrm{T}}{dp}=\frac{2a}{\mathrm{RT}}-b.$$

Following Joule and Kelvin, let us write
$$-\frac{d\mathrm{T}}{dp}=\frac{\theta}{\Pi},$$
whence we obtain
$$\theta=\frac{\Pi}{\mathrm{J}\,.\,\mathrm{K}_p}\left(\frac{2a}{\mathrm{RT}}-b\right),$$
$$=\frac{\alpha}{\mathrm{T}}-\beta,$$
where α and β are constants. This is exactly the formula proposed by Rose-Innes, from an examination of the results of the experiments of Joule and Lord Kelvin; it is not without interest to find that this formula is directly deducible from van der Waals's equation.

(3) The result obtained in the foregoing paragraph suggested the advisability of making a similar calculation on the basis of Clausius's characteristic equation
$$\left\{p+\frac{\mathrm{A}}{\mathrm{T}(v+\mathrm{C})^2}\right\}(v-\mathrm{B})=\mathrm{RT},$$
which admittedly represents the behaviour of the easily condensible gases with greater accuracy than van der Waals's equation does.

Dividing throughout, as before, by $v-\mathrm{B}$, and differentiating with respect to T (assuming $p=$constant), we obtain
$$-\frac{\mathrm{A}}{\mathrm{T}^2(v+\mathrm{C})^2}-\frac{2\mathrm{A}}{\mathrm{T}(v+\mathrm{C})^3}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p=\frac{\mathrm{R}}{v-\mathrm{B}}-\frac{\mathrm{RT}}{(v-\mathrm{B})^2}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p;$$
i. e.
$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p=\frac{\dfrac{\mathrm{R}}{v-\mathrm{B}}+\dfrac{\mathrm{A}}{\mathrm{T}^2(v+\mathrm{C})^2}}{\dfrac{\mathrm{R}}{(v-\mathrm{B})^2}-\dfrac{2\mathrm{A}}{\mathrm{T}^2(v+\mathrm{C})^3}}$$
$$=\left\{v-\mathrm{B}+\frac{\mathrm{A}}{\mathrm{RT}^2}\left(\frac{v-\mathrm{B}}{v+\mathrm{C}}\right)^2\right\}\left\{1-\frac{2\mathrm{A}}{\mathrm{RT}^2}\cdot\frac{(v-\mathrm{B})^2}{(v+\mathrm{C})^3}\right\}^{-1}.$$

Neglecting powers and products of the small quantities A, B, and C, this becomes
$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p-v=\frac{3\mathrm{A}}{\mathrm{RT}^2}-\mathrm{B}; \quad . \quad . \quad . \quad (\text{q. p.})$$

whence (by the same reasoning as in the previous discussion) we obtain

$$\theta = \frac{\gamma}{T^2} - \delta,$$

where γ and δ are constants—an expression which differs from that given by Rose-Innes in that it involves the square of the temperature instead of its first power.

It may not be uninteresting to compare the results furnished by this formula with the figures published by Rose-Innes: the comparison is here given for CO_2 only, seeing that the behaviour of air and hydrogen is expressed about as well by van der Waals's equation as by that of Clausius, and so the calculation in their case is hardly worth making.

Plotting the values of θ and $\frac{1}{T^2}$ for CO_2 as ordinates and abscissæ we obtain a remarkably good straight line; from it the following values were deduced,

$$\gamma = 4{\cdot}11 \times 10^5; \qquad \delta = 0{\cdot}88.$$

With these constants the values of θ were recalculated for the experimental temperatures, and the annexed table gives the results. It will be seen that, slight as are the differences between the results of observation and those furnished by Rose-Innes's formula, the differences in the case of the new formula are still smaller, as would naturally be anticipated from the above discussion.

Table.

Temp. Centigrade.	θ (observed).	θ (calculated) (Rose-Innes).	Differences (obs.—calc.).	θ (calculated) (new formula).	Differences (obs.—calc.).
0·0°	4·64	4·60	+0·04	4·64	0·00
7·4	4·37	4·35	+0·02	4·35	+0·02
35·6	3·41	3·49	−0·08	3·43	−0·02
51·0	2·95	3·02	−0·07	2·96	−0·01
93·5	2·16	2·16	0·00	2·18	−0·02
97·6	2·14	2·08	+0·06	2·12	+0·02

Several remarks are suggested by the discussion just given. In the first place, we observe that the Joule-Kelvin formula for the thermal effect would hold good for gases whose co-volume was zero, or negligible in comparison with the "molecular pressure"; it is not unlikely, indeed, that if the original investigators had plotted θ and $\frac{1}{T^2}$, instead of $\frac{\theta}{T^2}$ and T, they would have arrived at the formula here suggested. Be that as it may, their suggestion that the temperature enters into the expression in its second power is probably very near the truth.

Another point worth noticing is the testimony, borne by the calculations, to the remarkable accuracy of these experiments; this statement holds good whether Rose-Innes's formula or the present writer's be adopted, though most markedly, perhaps, in the latter case. This remark seems worth making as, in the present writer's opinion, too little credit has in the past been attached to the experiments in this particular; for example, Lehfeldt's statement*, that we probably do not know the value of θ for any single case within 10 per cent., seems to be a good deal wide of the mark.

Our discussion further supplies an answer to the critique on Clausius's formula expressed by Natanson † in his able memoir on the Joule-Thomson effect. He affirms that the results of the experiments render the existence of any physical quantity corresponding to Clausius's β—styled C in this paper—improbable. As a matter of fact, the experiments give no evidence at all on this point; for the disputed constant enters only into terms which involve the second order of small quantities, terms which would be insensible in any "porous plug" experiments hitherto performed.

(4) *Some Thermodynamical Consequences of Joule and Thomson's Experiments.*

(a) *The Relation between the Intrinsic Energy of ordinary Gases and their Volume.*—If a substance—whether gaseous or not—be allowed to expand infinitesimally without doing

* Phil. Mag. April 1898. † Wied. *Ann.* xxxi. p. 502.

work or gaining heat, as in a "porous plug" experiment, we know that

$$d(\mathrm{U}+pv)=0\,;$$

if at the same time its temperature falls through a range $d\mathrm{T}$, we must communicate to it an amount of heat (reckoned in ergs) $=\mathrm{J}\,.\,\mathrm{K}_v\,.\,d\mathrm{T}$ if we desire to restore the substance to its initial temperature without gaining or expending work.

Hence any infinitesimal isothermal change in the intrinsic energy may be calculated as follows :—

1st. Let the substance expand through a porous plug till its final volume is attained ; the gain of internal energy is $-(pdv+vdp)$ and the fall of temperature $d\mathrm{T}$.

2nd. Heat the substance at constant volume till its initial temperature is regained ; this increases the intrinsic energy by an amount $\mathrm{J}\,.\,\mathrm{K}_v\,.\,d\mathrm{T}$. Hence

$$d_{\mathrm{T}}\mathrm{U}=-pdv-vdp+\mathrm{J}\,.\,\mathrm{K}_v\,.\,d\mathrm{T}.$$

This is true for all substances ; to apply it to gases we proceed as follows :—

Writing
$$d\mathrm{T}=-\frac{\theta}{\Pi}dp,$$

as is done by Joule and Lord Kelvin, we obtain

$$d_{\mathrm{T}}\mathrm{U}=-pdv-\left(v+\frac{\theta}{\Pi}\,.\,\mathrm{J}\,.\,\mathrm{K}_v\right)dp\,;$$

whence
$$\left(\frac{\partial\mathrm{U}}{\partial v}\right)_{\mathrm{T}}=-\left\{p+\left(v+\frac{\theta}{\Pi}\,.\,\mathrm{J}\,.\,\mathrm{K}_v\right)\frac{dp}{dv}\right\},$$

where it is to be noted that $\frac{dp}{dv}$ is not a *partial* differential coefficient.

To evaluate this quantity we must determine $\frac{dp}{dv}$; and to do this we must assume some form for the gas-equation. It does not much matter which we take—as we shall see presently—for an approximation is all that is required ; anything further is rendered needless by the fact that $\left(\frac{\partial\mathrm{U}}{\partial v}\right)_{\mathrm{T}}$ is always small compared with $\left(\frac{\partial\mathrm{U}}{\partial\mathrm{T}}\right)_v$.

We will employ van der Waals's formula ; then

$$p=\frac{\mathrm{RT}}{v-b}-\frac{a}{v^2};$$

$$\therefore \frac{dp}{dv}=\frac{\mathrm{R}}{v-b}\frac{d\mathrm{T}}{dv}-\frac{\mathrm{RT}}{(v-b)^2}+\frac{2a}{v^3};$$

but $$\frac{d\mathrm{T}}{dv}=-\frac{\theta}{\Pi}\cdot\frac{dp}{dv}, \quad \text{and} \quad \frac{\mathrm{RT}}{v-b}=p+\frac{a}{v^2},$$

therefore

$$\frac{dp}{dv}\left(1+\frac{\theta}{\Pi}\frac{\mathrm{R}}{v-b}\right)=\frac{2a}{v^3}-\frac{p+\frac{a}{v^2}}{v-b},$$

i. e.

$$\frac{dp}{dv}=\left\{\frac{2a}{v^3}-\frac{p+\frac{a}{v^2}}{v-b}\right\}\left\{1+\frac{\theta}{\Pi}\frac{\mathrm{R}}{v-b}\right\}^{-1}.$$

Substituting this value of $\frac{dp}{dv}$ in the equation

$$\left(\frac{\partial \mathrm{U}}{\partial v}\right)_{\mathrm{T}}=-\left\{p+\left(v+\frac{\theta}{\Pi}\,.\,\mathrm{J}\,.\,\mathrm{K}_v\right)\frac{dp}{dv}\right\}$$

we obtain

$$\left(\frac{\partial \mathrm{U}}{\partial v}\right)_{\mathrm{T}}=\left\{v+\frac{\theta}{\Pi}\,.\,\mathrm{J}\,.\,\mathrm{K}_v\right\}\frac{\frac{p+\frac{a}{v^2}}{v-b}-\frac{2a}{v^3}}{1-\frac{\theta}{\Pi}\frac{\mathrm{R}}{v-b}}-p.$$

Since $\frac{a}{v^2}$ is in general small compared with p, and b and $\frac{\mathrm{R}\theta}{\Pi}$ are both small compared with v, we have

$$\left(\frac{\partial \mathrm{U}}{\partial v}\right)_{\mathrm{T}}=\left(v+\frac{\theta}{\Pi}\,.\,\mathrm{J}\,.\,\mathrm{K}_v\right)\frac{p}{v}-p \quad . \quad . \quad (\text{q. p.})$$

$$=\frac{\theta}{\Pi}\,.\,\frac{p}{v}\,.\,\mathrm{J}\,.\,\mathrm{K}_v$$

$$=\frac{\theta}{\Pi}\,.\,p\,.\,\rho\,.\,\mathrm{J}\,.\,\mathrm{K}_v.$$

We should have obtained the same expression if we had employed either the Boyle-Charles equation or that of

Clausius ; in the former case as the direct result, in the latter as an approximation of the same order as that made above.

We may now evaluate this quantity numerically for air at standard pressure and temperature ; for this purpose we have the approximate values $\theta=0\cdot92$; $p=\Pi$; $\rho=0\cdot00129$; $K_v=0\cdot17$; $J=4\cdot2\times10^7$; whence

$$\left(\frac{\partial U}{\partial v}\right)_T=8\cdot5\times10^3. \quad . \quad . \quad . \quad . \qquad \text{(q. p.)}$$

(b) *Ratio of the two Specific Heats of a Gas.*—We have for all bodies

$$K_v=\frac{1}{J}\left(\frac{\partial U}{\partial T}\right)_v; \quad K_p=\frac{1}{J}\left\{\left(\frac{\partial U}{\partial T}\right)_v+\left(\frac{\partial U}{\partial v}\right)_T\left(\frac{\partial v}{\partial T}\right)_p+p\left(\frac{\partial v}{\partial T}\right)_p\right\}$$

$$=K_v+\frac{\sigma}{J\rho}\left\{\left(\frac{\partial U}{\partial v}\right)_T+p\right\};$$

$$\frac{K_p}{K_v}=\frac{K_p}{K_p-\frac{\sigma}{J\rho}\left\{\left(\frac{\partial U}{\partial v}\right)_T+p\right\}}.$$

If, therefore, we make use of the value of $\left(\frac{\partial U}{\partial v}\right)_T$ as determined in the preceding subsection, we can determine this ratio for ordinary gases.

We give the result for air at standard pressure and temperature ; here $K_p=0\cdot2375$; $\sigma=0\cdot003665$; $\rho=0\cdot001293$; $J=4\cdot212\times10^7$*; $p=1\cdot0136\times10^6$; $\left(\frac{\partial U}{\partial v}\right)_T=8\cdot5\times10^3$; whence

$$\frac{K_p}{K_v}=1\cdot408.$$

This is exactly the value deduced from the velocity of sound ; it is also the mean of the results given by Röntgen (1·406), and by Jamin and Richard (1·41), who worked by totally different methods.

Had air been a perfect gas, $\left(\frac{\partial U}{\partial v}\right)_T$ would have been zero, and our formula would become

$$\frac{K_p}{K_v}=\frac{K_p}{K_p-p\,.\,\frac{\sigma}{J\rho}}.$$

* Obtained by extrapolation from Rowland's results.

In this case the same values of K_p, p, σ, and ρ would give $\frac{K_p}{K_v} = 1{\cdot}400$, which is the value assigned by the kinetic theory for a perfect gas consisting of diatomic molecules. The Joule-Thomson effect, then, completely accounts for the difference between the "observed" and "theoretical" values of this important ratio; and verifies the suggestion, made long ago, that the "imperfection" of the gas is probably the source of this difference.

(5) *The Maximum Density Point.*

Some interesting thermodynamic properties of substances at their temperature of maximum density are deducible from the equation, employed in the last section,

$$K_p - K_v = \frac{\sigma}{J\rho}\left\{\left(\frac{\partial U}{\partial v}\right)_T + p\right\};$$

as the present writer does not remember to have seen them stated anywhere, he ventures to insert them here. Lord Kelvin * has shown that the following equation holds good for all substances,

$$\frac{K_p}{K_v} = \frac{K_p}{K_p - T \cdot \frac{\sigma^2}{J\rho} \cdot e_T}$$

(where e_T denotes the isothermal elasticity, and the notation is changed to that used in this paper); hence we get

$$K_p - K_v = T \cdot \frac{\sigma^2}{J\rho} \cdot e_T;$$

combining this equation with the previous one we obtain

$$\left(\frac{\partial U}{\partial v}\right)_T = T \cdot \sigma \cdot e_T - p$$

for all substances.

If, then, a substance have a maximum density at any particular temperature, σ vanishes for that temperature, and we get

$$\left(\frac{\partial U}{\partial v}\right)_T = -p; \quad K_p = K_v.$$

But this is not all. The first law of thermodynamics,

* *Encyc. Brit.* art. "Elasticity."

written in its ordinary form, is

$$J \, . \, dQ = \left(\frac{\partial U}{\partial T}\right)_v . \, dT + \left\{\left(\frac{\partial U}{\partial v}\right)_T + p\right\} . \, dv;$$

at the maximum density point $\left(\frac{\partial U}{\partial v}\right)_T + p$ vanishes, as we have just seen ; consequently

$$J \, . \, dQ = \left(\frac{\partial U}{\partial T}\right)_v . \, dT.$$

If dT be zero dQ vanishes; in other words, *no heat is absorbed or evolved in any isothermal transformation at this temperature; i. e.* the latent heat of isothermal transformation is zero. Similarly dT vanishes if dQ be zero.

Many interesting conclusions may be drawn from this remarkable result; among them are the following :—

1st. The Joule-Thomson effect is zero for every substance at its maximum density, just as it is for an ideal perfect gas, though for a very different reason.

2nd. The *infinite number* of specific heats which every substance possesses is reduced at this point to *one.*

University of Melbourne,
December 5th, 1898.

Discussion.

Mr. Rose-Innes congratulated the author on having written an interesting paper on a difficult subject. At the same time he felt bound to acknowledge that he was out of sympathy with the general idea contained in the paper. The experimental difficulties that occurred in carrying out the Joule-Thomson measurements were so enormous that it was better to rely on them as little as possible, notwithstanding the great skill of the experimenters. The Joule-Thomson results could not be disregarded altogether, since they were necessary for the establishment of the thermo-dynamic scale ; but once that scale had been set up, it was better to have recourse as much as possible to such experiments as those of M. Amagat on the compressibility of gases. He also pointed out that one of the deductions given in the paper from van der Waals's formula had already been given by van der Waals himself.

XLV. *On the Criterion for the Oscillatory Discharge of a Condenser.* *By* E. H. BARTON, *D.Sc., F.R.S.E., and* W. B. MORTON, *M.A.**

THE ordinary condition for the oscillatory discharge of a condenser of capacity C through a wire of resistance R and inductance L, namely

$$C < \frac{4L}{R^2},$$

is obtained from the differential equation

$$0 = \frac{Q}{C} + R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

by making Q proportional to e^{xt} and expressing the condition that the resulting quadratic has imaginary roots. Now if account be taken of the distribution of the current in the wire, the well-known work of Maxwell and Lord Rayleigh † shows that the above differential equation must, for a straight wire, be supplemented by terms on the right,

$$-\tfrac{1}{12}R\alpha^2\mu^2\frac{d^3Q}{dt^3} + \tfrac{1}{48}R\alpha^3\mu^3\frac{d^4Q}{dt^4} - \tfrac{1}{180}R\alpha^4\mu^4\frac{d^5Q}{dt^5}, \text{ \&c.}, \quad . \quad . \quad (2)$$

where α is the length of the wire divided by its resistance $\left(\frac{l}{R}\right)$, and μ the permeability of the material of the wire. For a curved wire we should probably have an equation of the same *form* with different coefficients. The question is, how do these additional terms affect the condition for oscillatory discharge?

The added terms have coefficients which are in general small compared with R. Lord Rayleigh (*loc. cit.*) takes, for iron, the value 10^4 for resistivity and 300 for μ. This would give for a wire of radius a, $\mu\alpha = \frac{3\pi a^2}{100}$; so that the coefficient of $\frac{d^3Q}{dt^3}$ would be, even for thick wire, less than $R \times 10^{-6}$. For copper the value would be still less. If, therefore, we

* Read March 24, 1899.

† Maxwell, 'Treatise,' vol. ii. § 690; Rayleigh, Phil. Mag. vol. xxi. p. 381 (1886).

put e^{xt} for Q, we have an algebraic equation of which the terms above the second are of small and decreasing importance. The effect of these small terms will be (1) to introduce very large roots corresponding to very rapid oscillations; these will clearly be of small amplitude, and will not affect the main phenomena of discharge: (2) to modify the original roots of the uncorrected quadratic equation. The cases of oscillatory and non-oscillatory discharge are separated by the case of equality of these displaced roots.

It is easy to see that the effect of the added terms will be to make the critical value on the simple theory $\left(C=\frac{4L}{R^2}\right)$ correspond actually to an oscillatory discharge. For in this case the graph of $y=\frac{1}{C}+Rx+Lx^2$ evidently touches the axis of x at a point on the negative side of the origin, viz. $x=-\frac{R}{2L}$, and lies entirely above the axis. If we compound with this the graph of the additional terms,

$$y=-\tfrac{1}{12}R\alpha^2\mu^2x^3 \text{ \&c.},$$

which, for negative x, also lies above the axis, we get the parabolic graph displaced upwards, so that the roots of the modified equation become imaginary.

To find to any desired degree of approximation the condition for equality of the chief roots of the equation, we may use the principle that a repeated root of an equation $f(x)=0$ is a root also of the derived equation $f'(x)=0$.

In what follows we shall write the differential equation of discharge in the form

$$0=\frac{Q}{C}+R\frac{dQ}{dt}+L\frac{d^2Q}{dt^2}+\phi\left(\frac{d}{dt}\right)Q, \quad . \quad . \quad . \quad (3)$$

and the corresponding algebraic equation

$$0=\frac{1}{C}+Rx+Lx^2+\phi(x), \quad . \quad . \quad . \quad . \quad (4)$$

using ϕ for the series of small terms in $\alpha^2\mu^2$, &c.

The derived equation is

$$0=R+2Lx+\phi'(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let the common root of these equations be $x = -\frac{R}{2L} + \theta$, where θ is small; then we can find θ to any degree of approximation from (5), which becomes

$$0 = 2L\theta + \phi'\left(-\frac{R}{2L} + \theta\right)$$

$$= 2L\theta + \phi_0' + \phi_0''.\theta + \tfrac{1}{2}\phi_0'''.\theta^2 + \&c., \quad . \quad . \quad . \quad (6)$$

where ϕ_0 is put for $\phi\left(-\frac{R}{2L}\right)$.

This gives

$$\theta = -\frac{\phi_0'}{2L} + \frac{\phi_0'\phi_0''}{4L^2} - \frac{\phi_0'}{16L^3}(2\phi_0''^2 + \phi_0'\phi_0'''). \quad . \quad (7)$$

Equation (4) becomes, in terms of θ,

$$0 = \frac{1}{C} - \frac{R^2}{4L} + L\theta^2 + \phi\left(-\frac{R}{2L} + \theta\right). \quad . \quad . \quad . \quad (8)$$

Putting in this the value (7), we get the required condition in the form

$$\frac{1}{C} = \frac{R^2}{4L} - \phi_0 + \frac{\phi_0'^2}{4L} - \frac{\phi_0'^2\phi_0''}{8L^2}, \quad . \quad . \quad . \quad . \quad (9)$$

which goes to the third order in ϕ_0 or the sixth in $\alpha\mu$.

Inserting the numerical values and arranging in powers of $\alpha\mu$, it comes out

$$\frac{1}{C} = \frac{R^2}{4L} - \frac{1}{96}\frac{R^4\alpha^2\mu^2}{L^3} - \frac{1}{768}\frac{R^5\alpha^3\mu^3}{L^4} + \frac{37}{46080}\frac{R^6\alpha^4\mu^4}{L^5}. \quad . \quad (10)$$

Since $\alpha = \frac{l}{R}$, where l is the length of the wire, we have the correcting terms expressed as a series in $\frac{\mu l}{L}$ thus

$$\frac{1}{C} = \frac{R^2}{4L}\left\{1 - \frac{1}{24}\left(\frac{\mu l}{L}\right)^2 - \frac{1}{192}\left(\frac{\mu l}{L}\right)^3 + \frac{37}{11520}\left(\frac{\mu l}{L}\right)^4 \ldots\right\}. \quad (11)$$

Equations (10), (11) show that the critical capacity is greater than that given by the simple theory.

It is interesting to compare with the above another and more physical way of treating the question. In a paper by one of us* read before the Physical Society on January 27th of this

* Barton, Proc. Phys. Soc. *ante*, p. 400.

year, expressions were obtained for the equivalent resistance and inductance of a wire for damped simple harmonic oscillations. The method of that paper was equivalent to putting for Q in equation (3) the value $e^{(ip-kp)t}$ and arranging the result in the form

$$0=\frac{1}{C}+(ip-kp)R''+(ip-kp)^2L''. \quad . \quad . \quad (12)$$

The real quantities R″, L″ gave the resistance and inductance required. To apply this to the present problem,—we can evidently get an approximation to the criterion sought by replacing R and L in the ordinary formula $C=\frac{4L}{R^2}$ by modified values appropriate to the case. These may be got approximately by putting, in the expressions for R″ and L″, $p=0$ and $kp=\frac{R}{2L}$.

The result obtained by this method agrees with (11) as far as the terms there given. Let us now examine what the process sketched above really amounts to as a mathematical treatment of equation (4), and to what order of small quantities its approximation holds good.

First we must put $ip-kp$ in ϕ and arrange in accordance with (12). Using Taylor's expansion, we have

$$\phi(ip-kp)=(ip-kp)\,.\,\frac{\phi(ip-kp)}{ip-kp}$$

$$=(ip-kp)\left[\frac{\phi(-kp)}{-kp}+ip\,.\left\{\frac{\phi'(-kp)}{-kp}-\frac{\phi(-kp)}{k^2p^2}\right\}+\ldots\right]$$

$$=(ip-kp)\left[-\frac{2\phi(-kp)}{kp}-\phi'(-kp)+\&c.\right]$$

$$+(ip-kp)^2\left[-\frac{\phi(-kp)}{k^2p^2}-\frac{\phi'(-kp)}{kp}+\&c.\right];$$

all terms in the square brackets, with the exception of those written, vanish with p.

Now put $p=0$, $kp=\frac{R}{2L}$, and referring to the complete equation (4) we see that the "equivalent" R″ and L″ become

respectively

$$R_0''=R-\frac{4L}{R}\phi_0-\phi_0',$$

$$L_0''=L-\frac{4L^2}{R^2}\phi_0-\frac{2L}{R}\phi_0'.$$

It may be easily verified that these expressions agree with those got from equations (15) and (16) of the paper quoted, when the values zero and $\frac{R}{2L}$ are put for p and kp respectively.

We see now that the use of the condition

$$C=\frac{4L_0''}{R_0''^2}$$

is equivalent, mathematically, to writing equation (4) in the form

$$0=\frac{1}{C}+x\left\{R+\frac{2\phi(x)}{x}-\phi'(x)\right\}+x^2\left\{L-\frac{\phi(x)}{x^2}+\frac{\phi'(x)}{x}\right\},$$

putting for x inside the brackets the value $-\frac{R}{2L}$, and treating the equation as a quadratic with constant coefficients.

As regards the degree of approximation given by this method, if we expand $\frac{R_0''^2}{4L_0''}$, we get

$$\frac{1}{C}=\frac{R^2}{4L}-\phi_0+\frac{\phi_0'^2}{4L}+\frac{\phi_0'^2}{2R^2}(2L\phi_0+R\phi_0')\ldots,$$

which agrees with the accurate expansion in (9) up to and including terms of the *second* order in ϕ. Or expressing the condition in a series of powers of $\alpha\mu$, as in (10), the two series coincide up to and including terms in $\alpha^5\mu^5$.

March 2nd, 1899.

Discussion.

Prof. Lodge said that the result naturally to be expected of "throttling," viz., the increase of resistance, and decrease of self-induction, due to the current keeping to the outside of the conductor, would tend rather to damp out the oscillations than to favour them.

Prof. EVERETT observed that the equation was no longer a quadratic, and that the quadratic criterion as to whether the discharge was oscillatory or non-oscillatory did not hold. The paper appeared to be consistent with itself, and he considered that the authors had satisfactorily proved, in their discussion of the equation of current, that the effect of "throttling" was to increase the tendency towards the oscillatory mode of discharge.

Prof. LODGE admitted that the quadratic criterion did not hold; he thought it most likely that the authors, who evidently had gone into the matter with care, were right. At the same time he wished to call attention to the singular and unexpected character of their conclusion. If it turned out that it was correct, *i. e.*, that there was no slip in sign, it was a result upon which he would desire to congratulate them. The result for damped might easily be different from the result for maintained oscillations; but it was not clear what would happen at the exactly dead-beat condition.

XLVI. *Supplementary Note to Paper "On the Criterion for the Oscillatory Discharge of a Condenser"**. *By* Dr. E. H. BARTON *and* Prof. W. B. MORTON†.

IN the discussion which followed the reading of this paper before the Physical Society it was pointed out that the result obtained—viz., that on taking into account the distribution of the current in the wire a condenser having the critical capacity on the simple theory gives an oscillatory discharge—seems to be contradicted by the well-known fact that the resistance of a wire is greater, and its inductance less, for oscillatory than for steady currents. Both the increase of resistance and the decrease of inductance should favour a *non* oscillatory discharge. The explanation of the apparent paradox is to be found in the effect of the damping on the inductance. It was shown in the paper by one of us‡

* See page 465 *suprà*. † Read May 12, 1899.

‡ Barton, "The Equivalent Resistance and Inductance of a Wire to an Oscillatory Discharge:" read January 27th, 1899. Proc. Phys. Soc. *ante*, p. 409.

which has been already quoted, that the damping of the oscillations causes an increase in both R and L. When the damping is great and the frequency small, as in the neighbourhood of our critical case, what may be termed the *equivalent inductance* L″ becomes *greater* than the steady-current value. The investigation of the present paper shows that this increase of L outweighs the increase of R in its effect on the criterion for oscillatory discharge. It seems worth while to examine this effect of damping on the inductance a little more closely.

The formula for equivalent inductance for current of the form $e^{-kpt+ipt}$ is (eq. 16, *loc. cit.*)

$$L''=l\left[A+\mu\left(\frac{1}{2}+\frac{kp\alpha\mu}{6}-\frac{1-3k^2}{48}p^2\alpha^2\mu^2-\&c.\right)\right],$$

which shows that when the damping kp is sufficiently large compared with p, then $L''>l(A+\frac{1}{2}\mu)$, *i. e.* greater than the steady-current value. We can form an approximate notion of the stage at which this takes place by noticing that when $k=\frac{p\alpha\mu}{8}$ the small terms written above are all positive. If we take the numerical value for $\alpha\mu$ in case of iron, used in the beginning of the paper, viz. $\alpha\mu=\frac{3\pi a^2}{100}$, and put $a=1$ mm., we have roughly $\frac{1}{1000}$ as the value of $\alpha\mu$. This would make the necessary damping $k=\frac{p}{8000}$, or the logarithmic decrement per wave equal to $\frac{2\pi p}{8000}$ roughly.

Putting $p=2\pi n$, where n is the frequency, this would give as the ratio of one amplitude to the next of the same sign about $\epsilon^{\frac{n}{200}}$.

Since the decrease of L with maintained oscillations is due to the concentration of current near the surface of the wire, it is at once suggested that we must have, in the present case, when the damping becomes important, an *axis*-concentration of current. The following investigation, by the method of Maxwell, shows that this is the case. In the discussion in the 'Treatise,' vol. ii. Art. 689, Maxwell expresses the current-density w at distance r from the axis of the wire by his

equation (3), which, modified by the introduction of μ (see Lord Rayleigh, Phil. Mag. May 1886), reads

$$-\pi\mu w = T_1 + 4T_2r^2 + 9T_3r^4 + \ldots + n^2T_nr^{2n-2} + \ldots$$

The T's are functions of the time which are subsequently connected with each other by the equations (10) (again modified by the insertion of μ),

$$T_1 = \frac{\pi\mu}{\rho}\frac{dT}{dt}, \ldots T_n = \frac{\pi^n\mu^n}{\rho^n(n\,!)^2}\frac{d^nT}{dt^n},$$

where ρ is the resistivity.

T is then expressed in terms of the *total* current C by

$$\alpha\frac{dT}{dt} = -C + \tfrac{1}{2}\mu\alpha\frac{dC}{dt} - \tfrac{1}{6}\mu^2\alpha^2\frac{d^2C}{dt^2} + \&c.$$

The reduction of these equations enables us to express w in terms of C and its differentials with respect to the time, thus

$$\pi a^2 w = C - \mu\alpha\frac{dC}{dt}\left(\frac{1}{2} - \frac{r^2}{a^2}\right) + \mu^2\alpha^2\frac{d^2C}{dt^2}\left(\frac{1}{6} - \frac{1}{2}\frac{r^2}{a^2} + \frac{1}{4}\frac{r^4}{a^4}\right), \ \&c.$$

To apply this to damped oscillations, put

$$C = e^{-kpt}\cos pt;$$

the right-hand side of the equation last written then becomes

$$e^{-kpt}\cos pt\left[1 + kp\alpha\mu\left(\frac{1}{2} - \frac{r^2}{a^2}\right) + \alpha^2\mu^2(k^2p^2 - p^2)\left(\frac{1}{6} - \frac{1}{2}\frac{r^2}{a^2} + \frac{1}{4}\frac{r^4}{a^4}\right)\&c.\right]$$

$$+ e^{-kpt}\sin pt\left[p\alpha\mu\left(\frac{1}{2} - \frac{r^2}{a^2}\right) + 2kp^2\alpha^2\mu^2\left(\frac{1}{6} - \frac{1}{2}\frac{r^2}{a^2} + \frac{1}{4}\frac{r^4}{a^4}\right)\&c.\right].$$

If we put this into the shape

$$Ae^{-kpt}\cos(pt + \beta),$$

A will be a quasi-amplitude of the disturbance at distance r from the axis. Its value works out

$$1 + \tfrac{1}{2}kp\alpha\mu\left(1 - \frac{2r^2}{a^2}\right) + \alpha^2\mu^2\left[k^2p^2\left(\frac{1}{6} - \frac{r^2}{2a^2} + \frac{r^4}{4a^4}\right) + \tfrac{1}{8}p^2\left(\frac{r^4}{a^4} - \frac{1}{6}\right)\right]\&c.$$

Now if $kp = 0$ while p is finite, so that there are undamped oscillations, then the last term in the last bracket is the first (besides the initial unit term) which does not vanish; and we see that its value falls off as r decreases—showing the ordinary *surface* concentration. If, on the other hand, the

damping is so great as to make the second term of the series more important than those which follow, then the value of A will increase as r decreases from a—showing *axis*-concentration. The coefficient of $kp\alpha\mu$ changes sign at $r=\frac{a}{\sqrt{2}}$. Assuming the preponderance of this term, this means that in the neighbourhood of this value we pass from a greater value on the inner parts to a less value on the outer, than would correspond to a uniform distribution throughout the wire.

From general reasoning it seems clear that if we think of a rapidly damped disturbance propagated into a wire from its boundary; and if the alternations are slow enough to allow the currents to penetrate to the core, we should expect to find an axial concentration during the latter stages of the phenomenon.

It may be of interest in this connexion to mention another case in which alternation and decay act in opposite ways as regards inductance. If oscillatory currents are being kept up in a primary coil, it is well known that the presence of a secondary decreases the effective inductance. But suppose that we have a steady current existing in a primary, that we cut off the applied E.M.F., and allow the current to die away. Then the nearness of a secondary coil will cause it to decay less rapidly, which corresponds to an *increase* of the apparent inductance. The case is worked out in Prof. J. J. Thomson's 'Elements of Electricity and Magnetism,' p. 391. It is there shown that the presence of a secondary of resistance S and inductance N changes the exponential which expresses the decay of current in the primary from

$$e^{-\frac{R}{L}t} \text{ to } e^{\frac{-R}{L\left(1+\frac{RN}{LS}\right)}t}.$$

Discussion.

Dr. Lehfeldt said that Prof. Lodge had pointed out, at the reading of the paper, that the solution the authors obtained changed character at the critical resistance. As this point had not been considered in the note, he supposed that the change in character made no difference to the results obtained.

XLVII. *On the Effect of a Solid Conducting Sphere in a Variable Magnetic Field on the Magnetic Induction at a Point outside.* *By* C. S. WHITEHEAD, *M.A.**

THIS paper is an investigation of the magnetic induction at a point outside a solid conducting sphere when magnetic disturbances are taking place in the dielectric outside.

From the equations obtained, it is shown that when the sphere becomes an infinite plate and the inducing system consists of an alternating current in a circular circuit whose plane is parallel to the surface of the plate, we have for a point just outside the plate and on the axis of the circuit,

$$R_d = \frac{6\pi\gamma\mu d}{k_1 f^3}\,; \quad T_d = 0;$$

and for a point at a considerable distance from the axis and just outside the plate,

$$R_d = \frac{9\pi\gamma\mu d f^2}{k_1 \rho^5},$$

$$T_d = 3\left(2\pi\gamma d + \frac{\pi\gamma\mu}{k_1}\right)\frac{f^2}{\rho^4},$$

where R_d is the maximum value of the magnetic induction normal to the surface,

T_d the maximum value of the induction tangential to the surface,

f the radius of the inducing circuit,

d the distance of the inducing circuit from the surface of the plate,

ρ the distance of the point from the axis,

γ the maximum current in the inducing circuit,

μ the magnetic permeability of the plate,

σ its specific resistance,

$p = 2\pi \times$ frequency,

$k_1 = \left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}$.

From these equations, and with the values of p, μ, and σ

* Read April 21, 1899.

given in the paper, the value of R_d for a sea-water plate is 44 times as great as it would be for an iron plate, and more than 3000 times as great as it would be for a copper plate.

It is also seen that for the purpose of Induction Telegraphy the receiving coil should have its plane vertical and not horizontal to get the best effect; firstly because d must in practice be small compared with ρ, and hence R_d is small compared with T_d; and secondly because R_d varies inversely as ρ^5 while T_d varies inversely as ρ^4.

Let P, Q, R be the components of the electromotive intensity,

a, b, c the components of the magnetic induction,

σ the specific resistance of the sphere,

μ the magnetic permeability of the sphere.

Let P, Q, R all vary as $e^{\iota pt}$, where $p=2\pi\times$ frequency and $\iota=(-1)^{\frac{1}{2}}$.

Neglecting the polarization-current, P, Q, R (in the sphere) satisfy

$$\nabla^2 P=\frac{4\pi\mu}{\sigma}\frac{dP}{dt},$$

$$\&c. = \quad \&c.$$

Let
$$k^2=-\frac{4\pi\mu\iota p}{\sigma}=(1-\iota)^2 k_1^2,$$

where
$$k_1=\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}};$$

$$\therefore\ \nabla^2 P+k^2 P=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with similar equations for Q and R. We also have

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=0.$$

The most general solution of these equations is given by (*cf.* Recent Researches, art. 370), omitting the time factor,

$$P=\Sigma\left\{(n+1)f_{n-1}(kr)\frac{d\omega_n'}{dx}-nk^2r^{2n+3}f_{n+1}(kr)\frac{d}{dx}\frac{\omega_n'}{r^{2n+1}}\right\}$$

$$+\Sigma f_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\omega_n. \quad . \quad . \quad . \quad . \quad (2)$$

Q and R satisfy similar equations. ω_n', ω_n are arbitrary

solid spherical harmonics of degree n, and $f_n(kr)$ satisfies

$$\frac{d^2 f_n}{dr^2}+\frac{2(n+1)}{r}\frac{df_n}{dr}+k^2 f_n=0. \quad . \quad . \quad . \quad (3)$$

In the sphere r can vanish, and we must take that solution of (3) which does not become infinite when r is zero,

$$\left.\begin{aligned} \therefore\ f_n(kr) &= \left(\frac{1}{kr}\frac{d}{d\,.\,kr}\right)^n \frac{\sin kr}{kr} \\ \text{also}\quad \frac{1}{kr}\frac{d\,.\,f_n}{d\,.\,kr} &= f_{n+1} \\ k^2r^2 f_{n+1}+(2n+1)f_n+f_{n-1} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Now the currents induced in the sphere are entirely due to magnetic forces outside the sphere, therefore (Recent Researches, art. 319) there can be no radial currents in the sphere, and therefore the radial electromotive intensity must vanish.

Hence the origin being at the centre of the sphere, we must have

$$\frac{x}{r}\mathrm{P}+\frac{y}{r}\mathrm{Q}+\frac{z}{r}\mathrm{R}=0\,;$$

therefore from (2)

$$\frac{n(n+1)}{r}\{f_{n-1}(kr)+k^2r^2 f_{n+1}(kr)\}\,\omega_n'=0,$$

$$\therefore\ -\frac{n(n+1)(2n+1)}{r}f_n(kr)\,.\,\omega_n'=0\,;$$

therefore $\omega_n'=0$.

Hence, restoring the time-factor,

$$\left.\begin{aligned} \mathrm{P} &= \Sigma f_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\omega_n e^{\iota pt} \\ \mathrm{Q} &= \Sigma f_n(kr)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\omega_n e^{\iota pt} \\ \mathrm{R} &= \Sigma f_n(kr)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\omega_n e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

Now

$$\left.\begin{aligned} -\frac{da}{dt} &= \frac{dR}{dy} - \frac{dQ}{dz} \\ -\frac{db}{dt} &= \frac{dP}{dz} - \frac{dR}{dx} \\ -\frac{dc}{dt} &= \frac{dQ}{dx} - \frac{dP}{dy} \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

therefore from (5)

$$\left.\begin{aligned} a &= -\frac{e^{\iota pt}}{\iota p(2n+1)}\left\{(n+1)f_{n-1}(kr)\frac{d\omega_n}{dx} \right. \\ &\qquad\qquad \left. -nk^2r^{2n+3}f_{n+1}(kr)\frac{d}{dx}\frac{\omega_n}{r^{2n+1}}\right\} \\ b &= -\frac{e^{\iota pt}}{\iota p(2n+1)}\left\{(n+1)f_{n-1}(kr)\frac{d\omega_n}{dy} \right. \\ &\qquad\qquad \left. -nk^2r^{2n+3}f_{n+1}(kr)\frac{d}{dy}\frac{\omega_n}{r^{2n+1}}\right\} \\ c &= -\frac{e^{\iota pt}}{\iota p(2n+1)}\left\{(n+1)f_{n-1}(kr)\frac{d\omega_n}{dz} \right. \\ &\qquad\qquad \left. -nk^2r^{2n+3}f_{n+1}(kr)\frac{d}{dz}\frac{\omega_n}{r^{2n+1}}\right\} \end{aligned}\right\}. \quad (7)$$

Equations (5) and (7) hold in the sphere.

In the dielectric, neglecting the displacement-current, a, b, c may be derived from a potential; we may therefore assume

$$\left.\begin{aligned} a &= \left\{\frac{d\Omega_n}{dx} + \mathbf{a}^{2n+1}\frac{d}{dx}\frac{\Omega_n'}{r^{2n+1}}\right\}e^{\iota pt} \\ b &= \left\{\frac{d\Omega_n}{dy} + \mathbf{a}^{2n+1}\frac{d}{dy}\frac{\Omega_n'}{r^{2n+1}}\right\}e^{\iota pt} \\ c &= \left\{\frac{d\Omega_n}{dz} + \mathbf{a}^{2n+1}\frac{d}{dz}\frac{\Omega_n'}{r^{2n+1}}\right\}e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad (8)$$

where $\mathbf{a}$ = radius of the sphere, and Ω_n, Ω_n' are solid harmonics of degree n, to be determined from the nature of the inducing system.

Let R_s = component of the magnetic induction along the radius vector in the sphere.

R_d = the component in the dielectric.

T_s = component of the magnetic induction tangential to a curve at right angles to the radius vector in the sphere.

T_d = the component in the dielectric.

$$R_s = \frac{x}{r}a + \frac{y}{r}b + \frac{z}{r}c,$$

$$T_s = a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds};$$

and similarly for R_d and T_d.

Hence from (7) and (8) we find, after some reductions,

$$\left.\begin{aligned}
R_s &= \frac{n(n+1)e^{\iota pt}}{\iota pr} f_n(kr)\omega_n \\
R_d &= \frac{e^{\iota pt}}{r}\left\{n\Omega_n - (n+1)\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\Omega_n'\right\} \\
T_s &= -\frac{e^{\iota pt}}{\iota p}\left\{f_{n-1}(kr) + nf_n(kr)\right\}\frac{d\omega_n}{ds} \\
T_d &= e^{\iota pt}\left\{\frac{d\Omega_n}{ds} + \left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds}\right\}
\end{aligned}\right\} \quad . \quad . \quad (9)$$

The values of P, Q, R in the sphere are given by (5), their values in the dielectric may be found from (6) and (8). We obtain

$$P = \iota pe^{\iota pt}\left\{\frac{1}{n+1}\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\Omega_n - \frac{1}{n}\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\Omega_n'\right\}$$

$$Q = \iota pe^{\iota pt}\left\{\frac{1}{n+1}\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\Omega_n - \frac{1}{n}\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\left(\frac{\mathrm{a}}{r}\right)^{2n+} \Omega_n'\right\}$$

$$R = \iota pe^{\iota pt}\left\{\frac{1}{n+1}\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\Omega_n - \frac{1}{n}\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\Omega_n'\right\}$$

Let $\mathfrak{T}_s$ be the component of P, Q, R tangential to the same curve as before in the sphere.

$\mathfrak{T}_d$ the component in the dielectric.

Let $\frac{dx}{ds'}, \frac{dy}{ds'}, \frac{dz}{ds'}$ be the direction cosines of a curve perpendicular to both the radius vector and the curve ds.

$$\therefore \quad x\frac{dx}{ds'} + y\frac{dy}{ds'} + z\frac{dz}{ds'} = 0$$

$$\frac{dx}{ds}\cdot\frac{dx}{ds'} + \frac{dy}{ds}\cdot\frac{dy}{ds'} + \frac{dz}{ds}\cdot\frac{dz}{ds'} = 0.$$

Therefore

$$\frac{\frac{dx}{ds'}}{z\frac{dy}{ds} - y\frac{dz}{ds}} = \frac{\frac{dy}{ds'}}{x\frac{dz}{ds} - z\frac{dx}{ds}} = \frac{\frac{dz}{ds'}}{y\frac{dx}{ds} - x\frac{dy}{ds}} = \frac{1}{r}.$$

Hence

$$\left.\begin{aligned} \Sigma_s &= \epsilon^{\iota pt} r f_n(kr)\frac{d\omega_n}{ds'} \\ \Sigma_d &= \iota p \epsilon^{\iota pt} r\left\{\frac{1}{n+1}\frac{d\Omega_n}{ds'} - \frac{1}{n}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds'}\right\} \end{aligned}\right\}. \qquad (10)$$

Let Σ_s', Σ_d' be the components of P, Q, R along ds' in the sphere and dielectric respectively.

We find as before

$$\left.\begin{aligned} \Sigma_s' &= \epsilon^{\iota pt} r f_n(kr)\frac{d\omega_n}{ds} \\ \Sigma_d' &= \iota p \epsilon^{\iota pt} r\left\{\frac{1}{n+1}\frac{d\Omega_n}{ds} - \frac{1}{n}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds}\right\} \end{aligned}\right\}. \qquad (10^*)$$

At the surface of the sphere, the normal magnetic induction is continuous, and also the tangential magnetic force.

Therefore

$$\mathrm{R}_s = \mathrm{R}_d \text{ when } r = \mathrm{a},$$

$$\frac{1}{\mu}\mathrm{T}_s = \mathrm{T}_d \quad ,, \quad r = \mathrm{a}.$$

Therefore

$$\frac{n(n+1)}{\iota p} f_n(k\mathrm{a})\omega_n = n\Omega_n - (n+1)\Omega_n'$$

$$-\frac{1}{\mu\iota p}\{f_{n-1}(k\mathrm{a}) + n\ {}_n k\mathrm{a})\}\omega_n = \Omega_n + \Omega_n'.$$

We see that the first of these equations also arises from the continuity of the tangential electromotive intensity.

Eliminating ω_n,

$$(n+1)\Omega_n' = -\frac{(\mu n+\mu+n)f_n(ka)+f_{n-1}(ka)}{(\mu-1)nf_n(ka)-f_{n-1}(ka)}\,n\Omega_n. \quad (11)$$

Now

$$f_n(kr)=\left(\frac{1}{kr}\frac{d}{d.kr}\right)^n\frac{\sin kr}{kr}$$

$$=(-)^n\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(kr)^{-(n+\frac{1}{2})}J_{n+\frac{1}{2}}(kr),$$

where $J_{n+\frac{1}{2}}(kr)$ is Bessel's function.

Hence when kr is large,

$$f_n(kr)=(kr)^{-(n+1)}\left\{P_n\sin\left(kr+\frac{n\pi}{2}\right)+Q_n\cos\left(kr+\frac{n\pi}{2}\right)\right\};$$

where $$P_n=1-\frac{(n-1)n(n+1)(n+2)}{2\,!\,(2kr)^2}+\ldots.$$

$$Q_n=\frac{n(n+1)}{2kr}-\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{3\,!\,(2kr)^3}+\ldots,$$

the real part of k is to be positive, therefore $k=(1-\iota)k_1$.

Let kr be so large that we may take

$$P_n=1,\qquad Q_n=\frac{n(n+1)}{2kr};$$

therefore

$$f_n(kr)=(kr)^{-(n+1)}\frac{e^{k_1r}e^{\iota(k_1r+\frac{n\pi}{2})}}{2\iota}\left\{1-\frac{n(n+1)}{2\iota kr}\right\},$$

$$\therefore\quad \frac{f_{n-1}(kr)}{f_n(kr)}=e^{-\frac{\iota\pi}{2}}kr\frac{1-\frac{(n-1)n}{2\iota kr}}{1-\frac{n(n+1)}{2\iota kr}}$$

$$=-\iota kr\left(1+\frac{n}{\iota kr}\right).$$

$$\therefore\quad (n+1)\Omega_n'=\frac{\iota ka-\mu(n+1)}{\iota ka+\mu n}\,n\Omega_n$$

$$=\left\{1-\frac{\mu(2n+1)}{\iota ka}\right\}n\Omega_n. \quad (12)$$

$$\therefore \quad (n+1)\Omega_n' = \left\{1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}} + \iota\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right\} n\Omega_n.$$

$$\therefore \quad n\Omega_n - (n+1)\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\Omega_n'$$

$$= n\Omega_n\left[1 - \left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right\} - \iota\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right]$$

$$= n\Omega_n\xi_n(\cos\zeta_n - \iota\sin\zeta_n),$$

where

$$\tan\zeta_n = \frac{\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\frac{\mu(2n+1)}{2k_1\mathbf{a}}}{1 - \left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right\}}$$

$$\xi_n^2 = \left[1 - \left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right\}\right]^2 + \left\{\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right\}^2.$$

$$\therefore \quad \xi_n = 1 - \left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right\}.$$

Now $\quad \mathrm{R}_d =$ real part of $\dfrac{e^{\iota pt}}{r}\left\{n\Omega_n - (n+1)\left(\dfrac{\mathbf{a}}{r}\right)^{2n+1}\Omega_n'\right\}$

$$= \ldots\ldots\ldots \frac{e^{\iota pt}}{r}n\Omega_n\xi_n e^{-\iota\zeta_n}$$

$$= \frac{\xi_n n\Omega_n}{r}\cos(pt - \zeta_n).$$

$$\therefore \quad \mathrm{R}_d = \Sigma\frac{n}{r}\left\{1 - \left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left(1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right)\right\}\cos(pt - \zeta_n)\Omega_n. \quad (13)$$

Again,

$$\frac{d\Omega_n}{ds} + \left(\frac{\mathbf{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds}$$

$$= \frac{d\Omega_n}{ds}\left[1 + \frac{n}{n+1}\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right\} + \iota\,.\,\frac{\mu n(2n+1)}{(n+1)2k_1\mathbf{a}}\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\right],$$

$$= \eta_n\left(\cos\phi_n + \iota\sin\phi_n\right)\frac{d\Omega_n}{ds};$$

where

$$\tan\phi_n = \frac{\dfrac{\mu n(2n+1)}{(n+1)2k_1\mathrm{a}}\left(\dfrac{\mathrm{a}}{r}\right)^{2n+1}}{1+\dfrac{n}{n+1}\left(\dfrac{\mathrm{a}}{r}\right)^{2n+1}\left\{1-\dfrac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}}$$

$$\eta_n^2 = \left[1+\frac{n}{n+1}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1-\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}\right]^2 + \left[\frac{\mu n(2n+1)}{(n+1)2k_1\mathrm{a}}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\right]^2 :$$

$$\therefore\quad \eta_n = 1+\frac{n}{n+1}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1-\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\};$$

$$\mathrm{T}_d = \text{real part of } e^{\iota pt}\left\{\frac{d\Omega_n}{ds}+\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds}\right\}$$

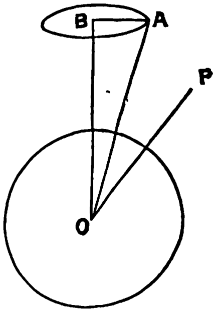
$$= \ldots\ldots\ldots e^{\iota pt}\eta_n e^{\iota\phi_n}\frac{d\Omega_n}{ds};$$

$$\therefore\ \mathrm{T}_d=\Sigma\left\{1+\frac{n}{n+1}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right)\right\}\cos\left(pt+\phi_n\right)\frac{d\Omega_n}{ds}. \quad . \quad (14)$$

In these formulæ (13) and (14), for R_d and T_d it is assumed that Ω_n is wholly real; this is the case in the application we are going to make.

Let the magnetic field outside the sphere be due to an alternating current in a circular circuit whose axis passes through the centre of the sphere.

Fig. 1.



Let O be the centre of the sphere,
B the centre of the circuit,
A any point on the circuit,
P any point.

Let BA $=f$; OA $=c$;
OP $=r$;
$\angle$BOA $=a$; $\angle$BOP $=\theta$.

Let Ω be the solid angle subtended by the circuit at P.
$\gamma e^{\iota pt}$ the current in the circuit,
V the potential due to the circuit at P,
U the magnetic force along OP.

Therefore
$$V=\gamma e^{\iota pt}\Omega,$$
$$U=-\frac{dV}{dr}=-\gamma e^{\iota pt}\frac{d\Omega}{dr}.$$

Now
$$\Omega=2\pi\left\{1-\cos\alpha+\sum_1^\infty\frac{\sin^2\alpha}{n}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta)\right\}.$$
$$r \angle c.$$

P_n is the zonal harmonic of the n^{th} order, and
$$P_n'=\frac{dP_n}{d\mu},\quad \mu=\cos\theta.$$
$$\therefore\quad U=-2\pi\gamma e^{\iota pt}\sum_1^\infty\frac{\sin^2\alpha}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta).$$

In the dielectric magnetic induction is the same thing as magnetic force.

Therefore, from the second equation of (9)
$$\Omega_n=-2\pi\gamma\frac{\sin^2\alpha}{n}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta).$$

Therefore
$$\frac{d\Omega_n}{rd\theta}=2\pi\gamma\frac{\sin^2\alpha}{n}\frac{\sin\theta}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n'(\theta).$$

Therefore, when $k\mathbf{a}$ is large,
$$R_d=-2\pi\gamma\sum_1^\infty\left[1-\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right)\right]$$
$$\times\frac{\sin^2\alpha}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta)\cos(pt-\zeta_n);\quad . . \text{(15)}$$

$$T_d=2\pi\gamma\sum_1^\infty\left[1+\frac{n}{n+1}\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right)\right]$$
$$\times\frac{\sin^2\alpha}{n}\frac{\sin\theta}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n'(\theta)\cos(pt+\phi_n),\quad . . \text{(16)}$$

if $ds=rd\theta$.

If r is but little greater than $\mathbf{a}$,

$$\tan\zeta_n=1, \quad \text{therefore } \zeta_n=\frac{\pi}{4} \text{ approximately;}$$

$$\tan\phi_n=0, \quad \text{therefore } \phi_n=0 \text{ approximately.}$$

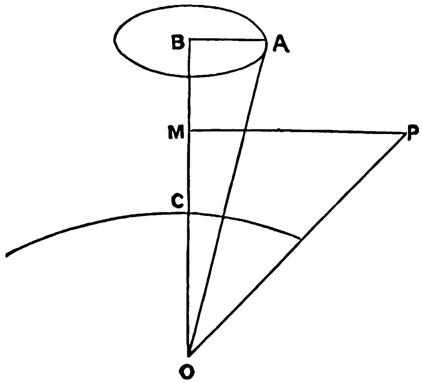
Therefore

$$R_d=-2\pi\gamma\cos\left(pt-\frac{\pi}{4}\right)\sum_1^\infty\left[\left\{1-\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right)\right\}\right.$$
$$\left.\frac{\sin^2\alpha}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta)\right];$$

$$T_d=2\pi\gamma\cos pt\sum_1^\infty\left[\left\{1+\frac{n}{n+1}\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right)\right\}\right.$$
$$\left.\times\frac{\sin^2\alpha}{n}\frac{\sin\theta}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n'(\theta)\right].$$

Let the radius of the sphere become infinite, the sphere becomes a plate, and we must change from spherical harmonics to Bessel's functions.

Fig. 2.



Let $\quad BC=d,$

$$BM=z, \qquad PM=\rho,$$

$$\sin\theta=\frac{\rho}{r}, \qquad \sin\alpha=\frac{y}{c}$$

Let $\quad n=\lambda r=\lambda c,$

where λ is to remain finite when n, r, and c become infinite.

$$\mathbf{a}=c-d, \qquad r=c-z \text{ ultimately};$$

$$\left(\frac{\mathbf{a}}{r}\right)^{2n+1}=e^{-2\lambda(d-z)}; \qquad \left(\frac{r}{c}\right)^{n}=e^{-\lambda z};$$

$$\mathrm{P}_n(\theta)=\mathrm{J}_0(\lambda\rho); \qquad \mathrm{P}_n'(\theta)=\frac{\lambda r^2}{\rho}\mathrm{J}_1(\lambda\rho);$$

$$\mathrm{P}_n'(a)=\frac{\lambda c^2}{f}\mathrm{J}_1(\lambda f).$$

$$\frac{\mu(2n+1)}{2k_1\mathbf{a}}=\frac{\mu\lambda}{k_1}, \quad rd\lambda=1.$$

Hence

$$\mathrm{R}_d=-2\pi\gamma f\int_0^\infty\lambda\left\{e^{-\lambda z}-\left(1-\frac{\mu\lambda}{k_1}\right)e^{-\lambda(2d-z)}\right\}\mathrm{J}_0(\lambda\rho)\mathrm{J}_1(\lambda f)\cos(pt-\zeta_n)d\lambda;$$

$$\mathrm{T}_d=2\pi\gamma f\int_0^\infty\lambda\left\{e^{-\lambda z}+\left(1-\frac{\mu\lambda}{k_1}\right)e^{-\lambda(2d-z)}\right\}\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda f)\cos(pt+\phi_n)d\lambda;$$

so that for a point just outside the plate,

$$\mathrm{R}_d=-\frac{2\pi\gamma f\mu}{k_1}\cos\left(pt-\frac{\pi}{4}\right)\int_0^\infty\lambda^2e^{-\lambda d}\mathrm{J}_0(\lambda\rho)\mathrm{J}_1(\lambda f)d\lambda, \quad . . (17)$$

$$\mathrm{T}_d=2\pi\gamma f\cos pt\int_0^\infty\left(2-\frac{\mu\lambda}{k_1}\right)\lambda e^{-\lambda d}\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda f)d\lambda. \quad . . . (18)$$

Now (*cf.* Gray and Mathews, 'Treatise on Bessel's Functions,' p. 27)

$$\mathrm{J}_0\sqrt{b^2+2bc\cos\alpha+c^2}=\mathrm{J}_0(b)\mathrm{J}_0(c)-2\mathrm{J}_1(b)\mathrm{J}_1(c)\cos\alpha$$
$$+2\mathrm{J}_2(b)\mathrm{J}_2(c)\cos 2\alpha-\ldots\ldots$$

Write $\lambda\rho$ for b and λf for c, multiply both sides by $e^{-\lambda d}$ and integrate with respect to λ from $\lambda=0$ to $\lambda=\infty$.

$$\therefore \quad \int_0^\infty e^{-\lambda d}\mathrm{J}_0(\lambda\rho)\mathrm{J}_0(\lambda f)d\lambda-2\int_0^\infty e^{-\lambda d}\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda f)d\lambda\cos\alpha$$

$$+\ldots.$$

$$=\int_0^\infty e^{-\lambda d}\mathrm{J}_0\{\lambda\sqrt{\rho^2+f^2+2\rho f\cos\alpha}\}d\lambda.$$

$$=\frac{1}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos\alpha}}$$

$$=\frac{1}{\pi}\int_0^\pi \frac{dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}}$$

$$+\frac{2}{\pi}\sum_1^\infty \cos n\alpha\int_0^\pi \frac{\cos nv\, dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}}$$

by Fourier's theorem;

$$\therefore \int_0^\infty e^{-\lambda d}\,J_0(\lambda\rho)J_0(\lambda f)\,d\lambda = \frac{1}{\pi}\int_0^\pi \frac{dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}} \quad . \quad . \quad (19)$$

$$\int_0^\infty e^{-\lambda d}\,J_n(\lambda\rho)J_n(\lambda f)\,d\lambda = \frac{(-)^n}{\pi}\int_0^\pi \frac{\cos nv\,.\,dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}} \quad (20)$$

Differentiate the first of these equations with respect to f. Since

$$\frac{dJ_0(\lambda f)}{df} = -\lambda J_1(\lambda f),$$

$$\int_0^\infty \lambda e^{-\lambda d}J_0(\lambda\rho)J_1(\lambda f)\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\pi \frac{(f+\rho\cos v)\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{3}{2}}} \quad . \quad . \quad (21)$$

Differentiate again with respect to d.

$$\therefore \int_0^\infty \lambda^2 e^{-\lambda d}J_0(\lambda\rho)J_1(\lambda f)\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\pi \frac{3d(f+\rho\cos v)\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}} \quad . \quad . \quad (22)$$

Differentiate (20) with respect to d.

$$\int_0^\infty \lambda e^{-\lambda d}J_n(\lambda\rho)J_n(\lambda f)\,d\lambda$$

$$=\frac{(-)^n}{\pi}\int_0^\pi \frac{d\,.\cos nv\,.\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{3}{2}}} \quad . \quad (23)$$

Differentiating again with respect to d,

$$\int_0^\infty \lambda^2 e^{-\lambda d}J_n(\lambda\rho)J_n(\lambda f)\,d\lambda$$

$$=\frac{(-)^n}{\pi}\int_0^\pi \frac{-(\rho^2+f^2+2\rho f\cos v-2d^2)\cos nv\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}}\,; \quad (24)$$

therefore, for a point in the dielectric just outside the plate,

$$\left.\begin{aligned} R_d &= -\frac{2\gamma\mu f}{k_1}\cos\left(pt-\frac{\pi}{4}\right)\int_0^\pi \frac{3d(f+\rho\cos v)\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}} \\ T_d &= -4\gamma f\cos pt\int_0^\pi \frac{d\,.\cos v dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{3}{2}}} \\ &\quad -\frac{2\gamma\mu f}{k_1}\cos pt\int_0^\pi \frac{(\rho^2+f^2+2\rho f\cos v-2d^2)\cos v\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}} \end{aligned}\right\} . \quad (25)$$

R_d is the magnetic induction perpendicular to the surface of the plate.

T_d the magnetic induction tangential to the surface along ρ.

Let the point be on the axis of the circuit, so that $\rho=0$. Therefore

$$\left.\begin{aligned} R_d &= -\frac{6\pi\gamma\mu}{k_1}\frac{df^2}{(d^2+f^2)^{\frac{5}{2}}}\cos\left(pt-\frac{\pi}{4}\right) \\ T_d &= 0. \end{aligned}\right\} \quad . \quad . \quad (26)$$

Let ρ be greater than f and large compared with d, so that d^2 may be neglected.

We have

$$(1+2h\cos v+h^2)^{-\frac{s}{2}} = (1+he^{\iota v})^{-\frac{s}{2}}(1+he^{-\iota v})^{-\frac{s}{2}}$$

$$=(1+A_1he^{\iota v}+A_2h^2e^{2\iota v}+\ \ldots\ldots\ +A_mh^me^{m\iota v}+\ \ldots)$$

$$\times(1+A_1h_1e^{-\iota v}+A_2h^2e^{-2\iota v}+\ \ldots\ +A_mh^me^{-m\iota v}+\ldots),$$

where

$$A_m=(-)^m\frac{s(s+2)\ \ldots\ (2m+s-2)}{2^m\,.\,m!}$$

$$A_0=1.$$

The coefficient of h^m

$$=2A_m\cos mv+2A_1A_{m-1}\cos(m-2)v+2A_2A_{m-2}\cos(m-4)v$$
$$+\ \ldots$$

the last term $=A^2_{\frac{m}{2}}$, if m be even

$=2A_{\frac{m-1}{2}}A_{\frac{m+1}{2}}\cos v$, if m be odd.

We hence find

$$\left.\begin{aligned} R_d &= \frac{9\pi\gamma\mu f^2 d}{k_1\rho^5}\cos\left(pt-\frac{\pi}{4}\right)\left\{1+\frac{1}{2}\left(\frac{5}{2}\right)^2\left(\frac{f}{\rho}\right)^2 \right. \\ &\qquad\left. +\frac{1}{3}\left(\frac{5\,.\,7}{2^2\,.\,2!}\right)^2\left(\frac{f}{\rho}\right)^4+\dots\right\} \\ T_d &= \left\{4\pi\gamma d+\frac{2\pi\gamma\mu}{k_1}\right\}\frac{f^2}{\rho^4}\cos pt\left\{\frac{3}{2}+\left(\frac{3}{2}\right)^2\frac{5}{4}\left(\frac{f}{\rho}\right)^2 \right. \\ &\qquad\left. +\left(\frac{3\,.\,5}{2^2\,.\,2!}\right)^2\frac{7}{6}\left(\frac{f}{\rho}\right)^4+\dots\right\} \end{aligned}\right\}. \quad (27)$$

Hence approximately when d is small compared with f and ρ large compared with f, we have, considering only maximum values, for a point on the axis

$$R_d=\frac{6\pi\gamma\mu}{k_1}\frac{d}{f^3};\ T_d=0, \quad . \quad . \quad . \quad . \quad (28)$$

and for a point a considerable distance from the axis

$$\left.\begin{aligned} R_d &= \frac{9\pi\gamma\mu}{k_1}\frac{f^2 d}{\rho^5} \\ T_d &= 3\left(2\pi\gamma d+\frac{\pi\gamma\mu}{k_1}\right)\frac{f^2}{\rho^4} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

It appears from these equations that when the primary circuit is near the surface of the plate the normal magnetic induction at a point just outside the plate is small, but that the tangential magnetic induction at a point some distance from the axis is not necessarily small; hence for the purpose of "Induction Telegraphy" the receiving-circuit should be vertical rather than horizontal.

Let $p=1885$, which makes the frequency 300; then for a plate in which $\mu=1$, $\sigma=2\,.\,10^{10}$, which are about the values for sea-water according to Mr. S. Evershed's experiments,

$$k_1=\cdot 00077;$$

for $\mu=1$, $\sigma=1600$, which are about the values for copper,

$$k_1=2\cdot 72;$$

for $\mu=1000$, $\sigma=10^4$, which are about the values for iron,

$$k_1=34\cdot 4;$$

for a sphere the size of the earth $\mathrm{a}=637\times 10^6$ cms.; so that in all these cases $k_1\mathrm{a}$ is large; hence for a sea-water plate the magnetic induction normal to the surface is 44 times as great as it would be were the plate of iron, and more than 3000 times as great as it would be were the plate of copper.

Let C_s' be the current in the sphere along the curve s'

$$C_s' = \frac{\Sigma'_s}{\sigma};$$

when $k\mathrm{a}$ is large,

$$\omega_n = \frac{\mu(2n+1)p}{(n+1)k\mathrm{a}}\,\frac{\Omega_n}{f(k\mathrm{a})};$$

therefore from 10 *,

$$C_s' = \frac{\mu p(2n+1)}{(n+1)k\mathrm{a}\sigma}\,\frac{rf_n(kr)}{f_n(k\mathrm{a})}\,\frac{d\Omega_n}{ds}\,e^{\iota pt}$$

$$= \frac{\sqrt{2}k_1(2n+1)}{4\pi(n+1)}\left(\frac{\mathrm{a}}{r}\right)^n e^{-k_1(\mathrm{a}-r)}\left\{1-\frac{n(n+1)(\mathrm{a}-r)}{4k_1\mathrm{a}r}\right\}$$

$$\times\cos\{pt-k_1(\mathrm{a}-r)+\chi\}\frac{d\Omega_n}{ds}, \quad \ldots \ldots \quad (30)$$

where $$\tan\chi = \frac{\dfrac{1}{2k_1}}{\dfrac{1}{2k_1}-\dfrac{n(n+1)(\mathrm{a}-r)}{4k_1^2\mathrm{a}r}}$$

Hence $\tan\chi=1$ approximately,

therefore $\chi=\dfrac{\pi}{4}$.

Hence when the inducing system is the circular circuit before mentioned, $ds=rd\theta$, and we find

$$C' = \frac{\sqrt{2}\gamma k_1}{2}\,e^{-k_1(\mathrm{a}-r)}\cos\left\{vt-k_1(\mathrm{a}-r)+\frac{\pi}{4}\right\}$$

$$\times\sum_1^\infty \frac{\sin^2\alpha\sin\theta}{r}\left(\frac{\mathrm{a}}{c}\right)^n\frac{2n+1}{n(n+1)}\left\{1-\frac{n(n+1)}{4k_1}\,\frac{\mathrm{a}-r}{\mathrm{a}r}\right\}P_n'(\alpha)P_n'(\theta). \quad (31)$$

Discussion.

Mr. Blakesley observed that, as a rule, experiment preceded theory. He congratulated Mr. Whitehead upon having settled from theoretical considerations that the vertical position of the receiving coil is best.

Prof. Everett said that the very elaborate method of analysis adopted in the paper appeared to be very clearly stated. He would like to know whether the inducing coil ought to be vertical as well as the receiving coil.

Mr. Appleyard thought that the experiment had left no doubt as to the best position of both coils. The early investigations of Mr. Willoughby Smith and the later work of Mr. H. R. Kempe and Mr. Preece had proved that for the best effect both coils should be vertical. But large vertical coils were difficult to fix and expensive to maintain. It was this reason probably that led Prof. Lodge to try what could be done with coils placed horizontally.

Mr. Whitehead, in reply, said that his final formulæ only applied to a horizontal inducing coil. He had not worked out the case of what would happen if the inducing coil itself was vertical. In the Flat Holm experiments both circuits were straight wires with their ends to earth, so that they really amounted to vertical coils.

XLVIII. *Note on the Vapour-Pressure of Solutions of Volatile Substances.* *By* R. A. Lehfeldt, *D.Sc.**

The change of vapour-pressure of a solvent due to the solution in it of a small quantity of volatile material has been calculated on the basis of Raoult's rule for the corresponding case of a non-volatile dissolved body, first by Planck, and later, more correctly, by Nernst; the formulæ obtained, and one experiment in support of them, are given in Ostwald's *Lehrbuch* (2. ii. 588). Further examples may be obtained from the measurements of vapour-pressures I have carried out on certain mixtures of organic liquids†; those measurements

* Read May 12, 1899.

† Phil. Mag. [5] vol. xlvi. p. 46.

were made primarily with the object of following out the behaviour of liquids in solutions not dilute ; but in the extreme cases the results obtained will serve as a test of the formulæ applying to dilute solutions.

Adopting the notation used in the memoir referred to, we have for a mixture of the two liquids A and B :—

π_A, π_B, vapour-pressure of A and B respectively,
p, „ mixture
ζ, ratio of number of mols. of A to mols. of B in liquid,
η, „ „ „ vapour.

Then the relation as expressed by Nernst becomes

$$\frac{\pi_B - p}{\pi_B} = \zeta - \frac{p}{\pi_B}\eta$$

for the case in which ζ is small, *i.e.* in which a small quantity of A is dissolved in B ; π_B is therefore the vapour-pressure of the solvent. This equation may more conveniently be transformed into

$$\frac{p}{\pi_B} = \frac{1-\zeta}{1-\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.})$$

When ζ, and consequently η, is very small, this becomes

$$\frac{p}{\pi_B} = 1 - \zeta + \eta,$$

the form in which Planck first gave the result, but which is less exact than the other when moderate concentrations are considered.

The expression in the above form (i.) is so simple that it is worth while to put it into words, thus :—

" *When a volatile substance is dissolved in a liquid the vapour-pressure of the liquid is altered in the ratio of the molecular fractional amount of solvent* (*Molenbruch*) *in the liquid to that in the vapour.*"

When the dissolved substance is non-volatile, the latter fraction becomes unity, and we recover the well-known formula $p/\pi_B = 1 - \zeta$.

It is perhaps worth remarking that the expression con-

sidered is a particular case of the thermodynamical relation applicable to any liquid mixtures, obtained by Duhem, Margules, and myself. The latter may be put * :—

$$(\eta-\zeta)\frac{\partial \log \frac{\eta}{1-\eta}}{\partial \log \frac{\zeta}{1-\zeta}} = \frac{\zeta(1-\eta)}{p}\frac{\partial p}{\partial \zeta},$$

whence

$$\frac{\eta-\zeta}{\eta(1-\eta)}\frac{\partial \eta}{\partial \zeta} = \frac{1}{p}\frac{\partial p}{\partial \zeta}.$$

If we integrate this for a small range from $\zeta=0$ and therefore $p=\pi_B$ we get

$$\frac{(\eta-\zeta)\eta}{\eta(1-\eta)} = \frac{p-\pi_B}{\pi_B},$$

or

$$\frac{p}{\pi_B}\eta-\zeta = \frac{p-\pi_B}{\pi_B},$$

the formula in question.

In case of a dilute solution of B in A it is easily seen that equation (i.) becomes

$$\frac{p}{\pi_A} = \frac{\zeta}{\eta} \qquad \text{(ii.)}$$

The experiments made were on four series of mixtures, viz. :—alcohol with benzene and toluene, and carbon tetrachloride with benzene and toluene; so that, taking the end members of each series, we get eight cases of "dilute solution." The second group of liquids, which were chosen as normal in character, give the following results :—

	η.	π (solvent).	p.	ζ (calc.).	ζ (obs.).
CCl_4 in toluene	0·181	93·0	105·4	0·072	0·073
Toluene in CCl_4	0·872	310·2	242·0	0·680	0·708
(A = carbon tetrachloride; B = toluene.)					
CCl_4 in benzene	0·054	270·9	273·6	0·045	0·043
Benzene in CCl_4.........	0·835	310·2	306·5	0·825	0·830
(A = carbon tetrachloride; B = benzene.)					

* *L. c.* p. 60.

Unfortunately no sufficiently dilute solution of toluene in CCl_4 was measured; that quoted contained (as may be seen from the value of ζ) 29·2 per cent. of the dissolved body, and it appears that the range of applicability of the formula is here exceeded: otherwise the agreement is good.

The mixtures containing alcohol, on the other hand, show maxima of vapour-pressures, and on this account the departure from the formulæ for dilute solutions is much more marked, and it is not possible to apply equations (i.) and (ii.) when the amount of dissolved body exceeds a very few per cent. The numbers are:—

	η.	π (solvent).	p.	ζ (calc.).	ζ (obs.).
Alcohol in benzene	0·281	270·9	350·4	0·070	0·088
Benzene in alcohol	0·668	219·5	315·0	0·958	0·886

(A = alcohol; B = benzene.)

Alcohol in toluene	0·591	93·0	199·5	0·123	0·138
Toluene in alcohol	0·902	219·5	233·5	0 959	0·946

(A = alcohol; B = toluene.)

Temperature, throughout = 50° C.

The quantity of dissolved substance in the liquid is less than that calculated, in each of the four alcohol mixtures. It is easy to show that for them the formula (i.) or (ii.) is less accurate than in the case of normal liquids, for a mixture in nearly equal parts may be looked upon either as a solution of A in B or of B in A. If it be calculated accordingly the results will of course differ, since the solution is not by any means dilute; but whilst in the case of the normal liquids the disagreement of the two results is small, in the alcohol mixtures it is egregious.

East London Technical College,
March 1899.

XLIX. *On the Thermal Properties of Normal Pentane.—Part II.* *By* J. ROSE-INNES, *M.A., B.Sc., and* Prof. SYDNEY YOUNG, *D.Sc., F.R.S.**

[Plate I.]

IN the first part of our paper on this subject, read before the Physical Society last December†, we published a large array of figures giving the relations between the volume, temperature, and pressure of normal pentane; it was suggested that a reasonably good agreement between calculation and experiment might be secured by using the formula

$$p=\frac{\mathrm{RT}}{v}\left\{1+\frac{e}{v+k-gv^{-2}}\right\}-\frac{l}{v(v+k)}$$

(*ante*, p. 336). This formula was first found by one of us for isopentane, but it seemed likely that it would do equally well for normal pentane; and arguments were adduced showing that in such case the values of R and of $l\div e$ might be taken to be the same for the two isomers. The question then arose whether l and e for normal pentane could be separately taken as equal to the values already found for isopentane, and the question was provisionally answered in the negative (*ante*, p. 337). Two methods were employed in dealing with this matter, but neither of them could be regarded as conclusive.

The first plan was to plot the differences of $\frac{1}{av^2}$ against $v^{-\frac{1}{3}}$; these differences grew smaller as the volume was made larger, but it did not seem likely that they vanished altogether when the volume was made infinite. The "wobbling" at large volumes, however, made it impossible to speak with any certainty one way or the other.

The second plan adopted was to take the three constants R, e, and l the same as for isopentane, and then to calculate suitable values for the two remaining constants k and g from the experimental data near the critical point. The isothermals

* Read May 26, 1899.

† Proc. Phys. Soc. present vol. p. 322; Phil. Mag. [5] vol. xlvii. p. 353.

were then calculated from the formula, and it was found that deviations occurred between calculation and experiment amounting to nearly 2 per cent. Accordingly the hypothesis on which the calculations were founded was considered to be most probably incorrect.

This second method of investigating the question has several advantages. We have to deal with the experimental data directly instead of indirectly as in the first method, and there is not any noticeable "wobbling" at large volumes. On the other hand, it is difficult to know how much of the deviation between calculation and experiment may be fairly attributed to experimental error, how much to the intrinsic imperfection of the formula, and how much to the special hypothesis employed. Further investigations were therefore undertaken, and the result of these forms the subject of the present communication.

After several ineffectual attempts, it was found possible to secure good concordance between calculation and experiment for volumes above 3·4 by means of the following hypothesis:— The values of R, of $l \div e$, and of g were taken as being the same respectively as those previously found for isopentane. The two constants l and k were given suitable values derived from the experimental data near the critical point. The values actually found were $l = 5{,}678{,}100$, $k = 3{\cdot}57$.

In order to test this hypothesis as thoroughly as possible a diagram was made in which pv was plotted against $v^{-\frac{1}{3}}$; the calculated isothermals were drawn as continuous lines while the experimental results were put in as dots; this diagram is reproduced on Plate I. An examination of it shows that the agreement between calculation and experiment is satisfactory; indeed the errors do not exceed 1 per cent. They are consequently less than those which occur in the tables published by one of us in conjunction with Prof. Ramsay as proving the truth of the linear law (Phil. Mag. xxiii. pp. 438–447). The improvement effected by means of the present hypothesis is so marked, that it is impossible to attribute it wholly or even chiefly to a compensation of errors: we may therefore regard our previous tentative hypothesis, that the value of l is the same for the two pentanes, as distinctly disproved. But we cannot assert, in the present state of the evidence, that our

present hypothesis is actually proved; we can merely note that it introduces errors so small as to be comparable with the intrinsic imperfection of the characteristic equation.

Finally we may conclude that the difference of pressure between two isomeric substances at the same temperature and volume involves the same power of the density as the first deviation from Boyle's law; *i. e.*, the second power.

Discussion.

Prof. Callendar expressed his interest in the wide applicability of the author's formula, and asked if any theoretical significance could be assigned to the various constants which appeared.

Mr. Rose-Innes said the R of their formula was the R of the perfect gas equation, and that the l and e corresponded respectively to the β and α of the ordinary van der Waals expression. So far as he knew the k and g had no physical interpretation.

L. *The Minor Variations of the Clark Cell.* *By* A. P. Trotter.*

[Abstract.]

A number of Clark cells was adopted three years ago, together with a resistance-box, as an electrical standard for the Colony of the Cape of Good Hope. The sum provided for the equipment of the Government Electrical Laboratory was a small one, and this, coupled with the risk of damage to the instruments during the journey, led to the adoption of the potentiometer (Crompton type), with various auxiliary resistances, in preference to balances or other instruments containing mechanism.

The cells consist of seven Muirhead portable Clark cells; the Board of Trade pattern with a considerable quantity of loose mercury not being considered to be so suitable. It was intended to compare the cells together periodically, and

* Read March 24, 1899.

to assume that if they kept in fair agreement they kept also fairly constant. It is intended also from time to time to supplement them by new cells. In addition to these Muirhead cells there are two Clark cells by Wolff with Reichsanstalt certificates, and two Carhart cells by Nalders.

Comparisons were made about once a month from July 1896 to February 1897, and then for about a year comparisons were made weekly. During the week ending April 3, 1897, comparisons were made twice a day. The object of this communication is to record the result of these comparisons, and to discuss the utility of the Clark cell as a laboratory standard.

The degree of precision aimed at is intermediate between that of high-class scientific measurements which are not likely to be required of the Government Electrical Laboratory of this Colony, and the commercial standard of 1 *in* 400 *or* ¼ *per cent.* The fact that the value of the Clark cell is not known absolutely with greater accuracy than 1 in 1000, has no more to do with the precision of these comparisons than the discrepancy of 0·018 mm. between the metre and the ten-millionth part of the quadrant has to do with the precision of ordinary measurements of length.

The interesting researches on the temperature-lag of Clark-cells*, and the considerable number of modifications of inverted†, dry‡, chloride§, and unsaturated || cells, which aim at reducing the temperature-coefficient or rendering it negligible for industrial purposes, have but little bearing on the use of the Clark cell for a laboratory standard; for the temperature of the cells can be easily kept sufficiently uniform to render the lag inappreciable. Whether the temperature coefficient be large or small is of no importance whatever, so long as it is definite.

The temperature of the Laboratory, when temporarily situated

* Ayrton and Cooper, Proc. Roy. Soc. vol. lix.; Electrician, xxxviii. p. 303; Spiers, Twyman, and Waters, Proc. Phys. Soc. xvi. p. 38.

† Callendar and Barnes, Rep. British Assoc. 1897; Electrician, xxxix. p. 638.

‡ Kahle, *Instrumentenkunde*, 1893, p. 298.

§ Hibbert, Electrician, xxxvii. p. 320.

|| Carhart, Electrician, xxiv. pp. 271, 643.

in the South African College, varied from about 28° to 13° C. during 1896. The cells are placed in a wooden box within a larger box, a space of three inches between being filled with loosely packed waste telegraph-slip. It had been intended to fill this space with aerated paraffin-wax or some other non-conductor; but no further precaution seemed necessary in view of the ultimate destination of the cells. At the South African College, a recording thermometer on the table showed daily variations of about 2° to 2½° C., and when placed under the table on the box, about 0°·75 C. The gradient at 10 A.M., when the comparison was generally made, had been about 0°·03 C. per hour for the previous 16 hours. Inside the box the changes were considerably less, and being considered negligible, were not investigated.

The Laboratory is now situated in the basement of the General Post Office; the box is placed below the floor in a small well. A thermometer, graduated in tenths of a degree, is lowered through a hole into the double box. Having regard to the precision of absolute measurement aimed at, and the various minor corrections which were disregarded, it was not attempted to read it closer than to tenths of a degree. Although the temperature was recorded in all cases as it was read, the slide of the potentiometer was sometimes set to the nearest round number on the main slide (scale of volts), if not more than about one-tenth of a degree from the ascertained temperature. This enabled a check reading, which was taken between each comparison, to be effected more quickly. The temperature-scale of the potentiometer is not divided to tenths of a degree, these were estimated.

A recording thermometer placed on the double box, in the well, shows that the temperature rarely varies more than ⅓ of 1° C. in 24 hours. The changes within the box are, of course, much less.

Since the mean of such Clark cells as agree well together is assumed to be the standard, it was necessary, in making the comparisons, to select an arbitrary standard and to compare each cell in turn with it. Cell No. 1 was thus used. In consequence of the preliminary balancing (after a rough balance had been obtained with the "hack" cells), and the

check test of No. 1 for steadiness of potentiometer current between each measurement of a cell, No. 1 was used from twice to three times more than the others. It does not, however, appear, from comparison between this cell and the mean of the whole set, to have been affected appreciably. Cell No. 1 was assumed to have the value given on the potentiometer-scale according to the engraved scale of temperature; viz. 1·4396 at 10° C., 1·4342 at 15° C., 1·4286 at 20° C., and so on. The scale of the potentiometer was then assumed to be a scale of volts, and the other cells were measured on it.

As the difference between the cells rarely exceeds 0·001 volt, only a quarter of an inch of the slide-wire, which corresponds to this difference, was used during any one comparison. The slide-wire has been carefully examined. An additional slider was made, and various lengths of the wire were intercepted between them. The greatest error (for purposes of comparison) would have been 1·47993 instead of 1·48, or 4·7 parts in 100,000; but there was no appreciable error at the part of the wire used. In making the comparisons the readings were taken generally to the fourth decimal figure, or one-tenth of a millivolt; the $\frac{1}{20}$ and even the $\frac{1}{40}$ of a millivolt was occasionally estimated. Mr. W. L. Logeman, a student at the South-African College, rendered much assistance in these comparisons as well as in other work in the Laboratory. Independent comparisons of the cells by Mr. Logeman and by myself very seldom varied in the fourth decimal figure, and often agreed even to the estimated halving ($\frac{1}{20}$ millivolt) of a division.

It was noticed that some of the cells, particularly the large Wolff cells, gave, for a given displacement of the slider, greater galvanometer-deflexions than others. This was obviously due to differences of internal resistance, and this led to the regular measurement of the internal resistances whenever the cells were compared. The mode adopted was to displace the slider 5 divisions, so as to cause a want of balance of 0·0005 volt, and to observe the deflexion. The deflexion was the same whether the slider was so displaced as to allow the cell to discharge, or whether it received a small charge shunted from the potentiometer current. The deflexion was

inversely as the sum of the resistances of the cell and the galvanometer. A scale of resistances empirically graduated was attached to the galvanometer-scale. The resistance of the Wolff cells was too low to be measured with any precision in this way.

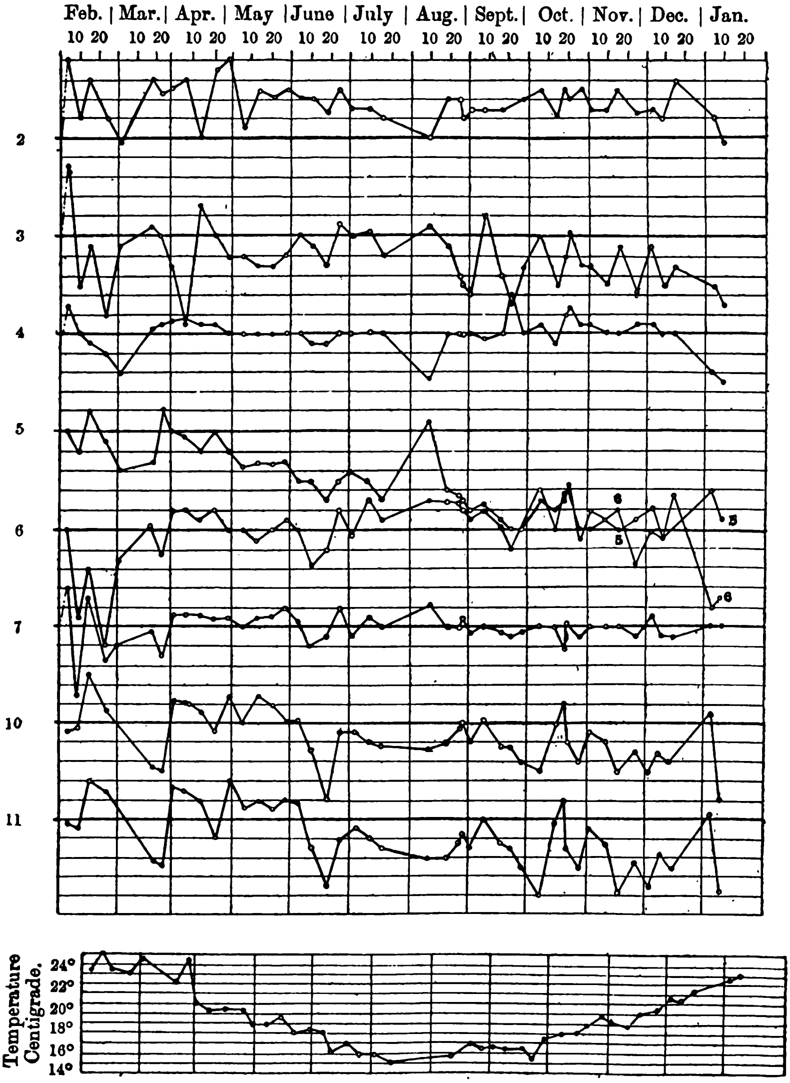
The results of the comparisons in 1897–98 are plotted in figs. 1 to 4. Fig. 1 shows the differences from cell No. 1. The heavy horizontal lines are zeros, and are numbered 2, 3, 4, &c., corresponding to the different cells. (Nos. 8 and 9 omitted, relate to the Carhart cells, which are not fairly comparable with the others.) The thinner horizontal lines represent differences of 0·0002 volt. The first observation is in each case joined to the corresponding zero line by a chain-dotted line, except in 5, 6, and 11, where little or no difference was found. Below this diagram is given a curve of the temperature of the cells. The most striking feature of this diagram is the close resemblance between the differences of the two large Wolff cells, Nos. 10 and 11; but a study of the lines representing the behaviour of the other cells leads at once to the question, What was No. 1 doing all this time?

In fig. 3 the differences from the mean of the nine cells are plotted. Here No. 1 appears on the same footing as the others. Nos. 10 and 11, while retaining their resemblance, are somewhat altered in detail. The first idea which occurs is that these variations are chiefly due to errors of observation and to minor corrigenda. But they are of quite another order. The probable error of setting and reading the slide of the potentiometer is about $\frac{1}{20}$ millivolt, represented by one quarter of the distance between two horizontal lines in the diagrams. Thermoelectric and plug-resistance comparison-errors, which must be distinguished from errors of measurement, are eliminated by the use of a null method.

The variations being so capricious, the cells were compared twice a day for a week, with the result shown in fig. 2. Even at such short intervals there is a lack of continuity in most of the curves; but the reduction in the variations between any two successive measurements is noteworthy.

The remarkable agreement of the Wolff cells together strongly suggests that these should be the standards. On the other hand, the curves 10 and 11, fig. 3, show that they

Fig. 1.—Differences from No. 1. One Division = 0·0002 volt = $\frac{1}{5}$ millivolt.



differ considerably from the mean of the other seven cells, for the mean on which fig. 3 is based is a mean weighted with these two cells themselves. They often differed by only a tenth of a millivolt, and seldom by more than a fifth of a

Fig. 2.—DIFFERENCES FROM No. 1.
ONE DIVISION=0·0002 volt.

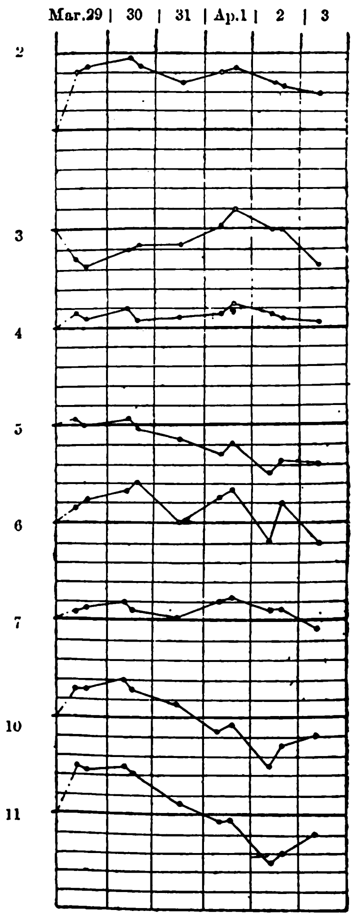
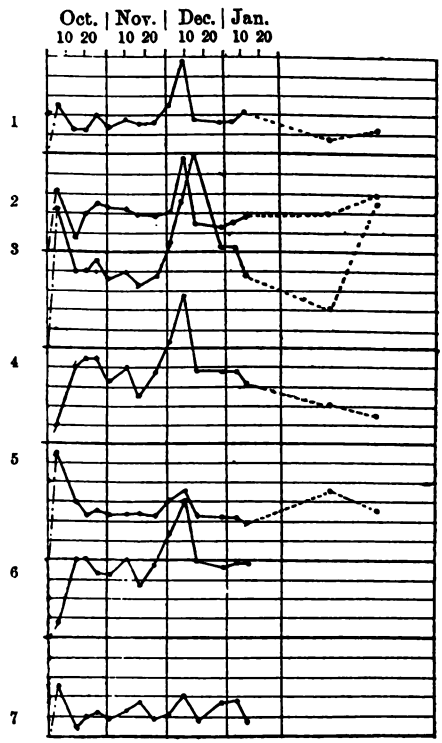


Fig. 4.—INTERNAL RESISTANCE.
ONE DIVISION=1000 ohms.

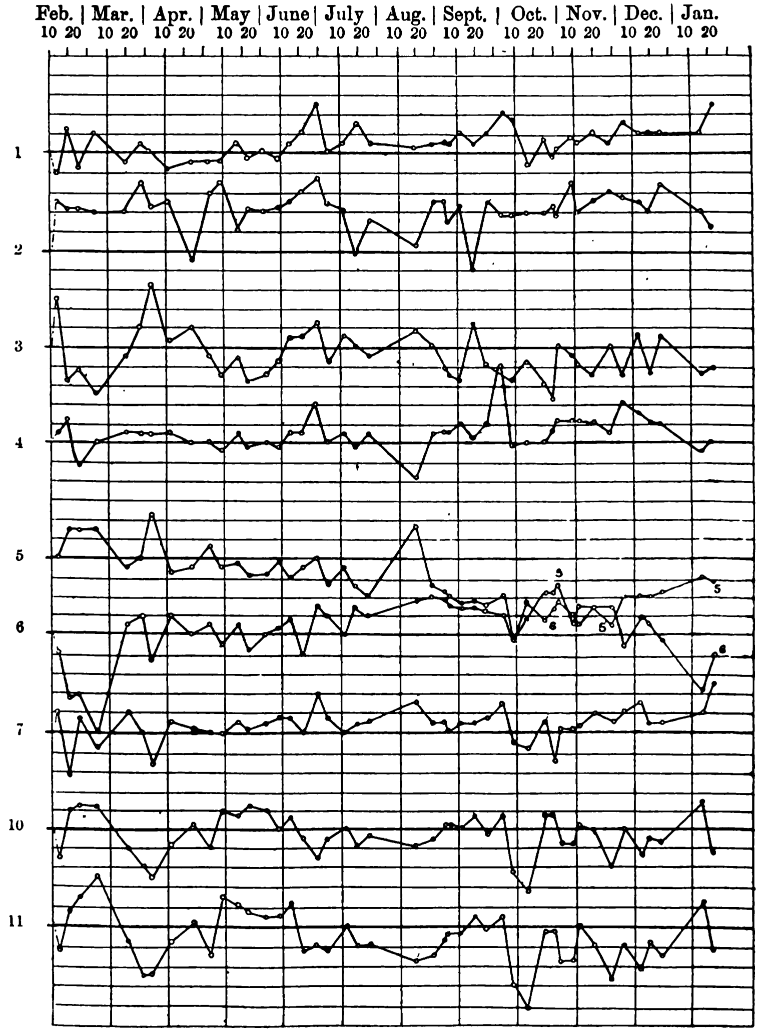


millivolt from each other, but on several occasions by more than half a millivolt from the mean of the whole.

A study of the variations of the other cells from the mean shows several runs of coincidence: for example, Nos. 1, 2, 4,

and 7 in June, Nos. 1, 4, and 6 in May. On the other hand, the contradictory variations are quite as well marked: for example, 3 and 5 with 6 and 7 in March, or 2 and 4 with

Fig. 3.—Differences from the Mean. One Div. =0·0002 volt=⅕ millivolt.



3 and 5 during the interval which unfortunately occurred between July 16 and August 10th. The marked drop in Nos. 10 and 11 at the end of September is found also in

No. 7, and to a less extent in 4, 5, and 6 ; but for the majority of the variations no coincidences can be discerned. There appears to be sufficient continuity in the curves to suggest that the internal state of the cells is by no means so quiescent as might be imagined. Perhaps the only general result is the falling off of No. 5, but no long series of comparisons was needed to show that.

The ease with which the internal resistance of the cells could be measured was only discovered in October, and so far as the observations went, no relation is apparent between the voltage and the resistance. The result is plotted in fig. 4. It is probable that the measurement on December 9 was faulty, but it must be remarked that there was an upward tendency among all the cells, especially 4 and 6. Hot weather was beginning, and there had been a slight fall of temperature, but no change sufficient to account for the discontinuity.

So far from being able to offer an explanation of these variations, I am unable to make any generalization about them or even to give a systematic description of them.

Two opportunities have occurred for comparison with newer Clark cells. One of these was 0·4 millivolt, and the other 0·6 millivolt lower than the mean of the standards. It is not improbable that this mean is now somewhat too high, as the cells are more than three years old; and there appears to be a tendency for Clark cells to rise a little in the course of time. *It should be observed, however, that the value of the mean of these cells in absolute measure is not the subject of the present communication.*

The capriciousness of the variations which have been described have a fortunate result, in that they appear to be on the whole as likely to be above as below the mean. The approximate probable difference of any cell from the mean is 0·23 millivolt; and the approximate probable deviation of the observed mean of the nine cells from the true mean is 0·077 millivolt or 5·4 parts in 100,000.

It appears, therefore, that the Clark cell used under conditions which may be easily attained in any permanent laboratory is of great value, at all events as a secondary standard, and that its value is worthy to be known to 1 part in 10,000 instead of 1 in 1000.

Discussion.

Mr. E. H. Griffiths said that the paper appeared to have value only in so far that it showed that Clark cells at Cape Town behaved in a manner that agreed with common knowledge and general experience everywhere else. Their variations depended upon shifts of temperature, and the consequent changes in the degree of saturation of the liquid. His own experiments during seven years, upon 42 Clark cells, had shown that if the temperature was kept constant to within 0·01° C., the steadiness and uniformity of all the E.M.F.'s was most remarkable. They started with discrepancies, which diminished with lapse of time until they became negligible. It was of little use to put a thermometer anywhere but within the cells; very slight changes of temperature caused serious changes in the degree of saturation of the liquid. The existence of the capricious lag of E.M.F. behind temperature precluded the possibility of formulating a satisfactory temperature correction for the ordinary form of Clark cells. In the case of Callendar cells there was no lag; the E.M.F. varied with temperature, by a definite amount, which could be corrected by a coefficient.

Mr. W. R. Cooper said the method of comparison used by the author was unsuitable, because to arrive at the differences of E.M.F. necessitated the measurement of the E.M.F. of each cell. The variations only amounted to a few ten-thousandths of a volt. The length of potentiometer-wire corresponding to a thousandth of a volt was only a quarter of an inch; under such conditions it would be difficult to ensure accuracy. A method of opposition would have been preferable. Mr. Cooper had found that Board of Trade cells only vary about one ten-thousandth of a volt between themselves from day to day under ordinary conditions. Cells of the H form vary about one-fifth of that amount.

Mr. A. P. Trotter (communicated). It appears that my desire to investigate with care the precision yielded by the simple means and apparatus which I have chosen, has been confused with a desire to arrive at a high degree of precision; a result which is sometimes important but often useless.

The degree of precision which was aimed at has been over-

looked in the discussion. I consider that the use of thermostats, water-baths, stirrers, ice, separate thermometers in each cell, and other refinements with which I am familiar, would, under the present circumstances, be quite out of place. I have the greatest regard for high precision in its proper place, as for example in the researches of Mr. Griffiths, but an accuracy of one part in a thousand is far in excess of any practical requirements of my official work. The whole equipment of the Government Electrical Laboratory, including storage-battery and testing sets for outdoor inspections, had to be provided for the sum of £300. I supplemented this with instruments and new purchases of my own. I consider that the potentiometer which I use is amply accurate for the requirements of this laboratory. It is probable that the correction of temperature errors of the potentiometer, thermoelectric and contact errors are quite as important as the use of a thermometer in each cell, and none of these are warranted by the circumstances. To adopt some of the refinements which have been suggested would, in this case, be as unintelligent as the use of seven-figure logarithms for work in which the data are only known to 2 or 3 per cent.

It is true that no precautions were taken to keep the temperature of the selected cell constant. So to use a comparison cell would be to make a standard of it. Why, then, have the other eight? My method is to use all the nine cells, treating none as being more worthy of being standards than others. The method is unusual, but my circumstances are unusual. I made no *measurements* of potential difference of the cells, for I have no other standard than the mean of the cells. I prefer to call this a comparison rather than a measurement. Having arrived at the mean, I am prepared to measure other differences of potential by it.

Bearing in mind that my object was a comparison, and not a measurement of a potential difference in terms of the length of a wire, details of the area of the slider-contact are of small importance, but as they have been asked for, I give them.

By slider-contact is probably meant the length of wire in contact with the slider, for the area has nothing to do with the matter. The brass knife-edge has become worn with use, and there is now a groove about $\frac{1}{3}$ mm. long axially,

and of a considerably flatter curvature than that of the wire, which is about 0·2 mm. in diameter (it is not worth while to disturb it to measure it exactly). The contact is made by the ingenious "let off" key of the Crompton potentiometer, with a definite pressure of between 1 and 2 grammes, irrespective of the pressure on the key. In making a measurement of the electromotive force of a cellin terms of the length of a wire, a knife-edge would be advantageous; but the limits of experimental error of the whole apparatus may be represented by a length of the wire; and any width of knife-edge considerably less than that length is a useless refinement. The present paper, however, does not discuss measurements of electromotive force, but of differences between cells. For that purpose, the length of wire in contact with the slider may be very much larger than in the former case without affecting the accuracy of the comparison. Similarly, since the temperature-coefficient is a linear function (within my limits of temperature), a mistake of a whole degree in reading the thermometer would not appreciably affect the comparisons.

I carefully considered the method of comparing the cells in pairs back to back, and decided, for reasons which need not be discussed here, that for the purpose of obtaining the mean of the whole set with the precision desired, the potentiometer method was better.

Intentionally, simple conditions were used; such conditions as may easily be realized in the test-room of engineering works, or in other Colonial laboratories; and the precision *of comparison* actually obtained, under these conditions, say 6 in 100,000, appears to be very satisfactory, and is much higher than I expected.

The allusion to the paper by Ayrton and Cooper on the "Variations in the Electromotive Force of the Clark Cell with Temperature," was accompanied by the remark that this and certain other papers have but little bearing on the use of the Clark cell for a laboratory standard. In the research recorded in that paper, the cells were taken through cycles of about 10° in 2½ hours, and only one small and slow cycle of 2° at the rate of 1° in 50 min. is described. These

are of interest in the case of portable cells carried about for inspections &c., or for cells left on a laboratory table with no temperature precautions. The temperature range at the South African College was 1° during 24 hours, and the temperature gradient was 0°·02 per hour for about 18 hours previous to the comparison. Such a gradient can be easily obtained in a more temperate climate than that of Cape Town.

No generalization is made in Ayrton and Cooper's paper on the connection between the magnitude of the lag and the temperature gradients. Only two gradients are given, viz., 4° per hour and 1·2° per hour. The difference between the rising and the falling volts appears to be about 15 and 4 ten-thousandths of a volt respectively, or, say, 2·8 millivolt per 1° per hour. From only two cases it cannot be determined whether the lag is simply proportional to the gradient; but assuming that that is the case, a gradient of 0·°02 per hour would give a lag of 0·056 millivolt, or an error of 0·028 millivolt, assuming the true value to lie midway between the rising and the falling curve. I went into this calculation at an early stage in the work, but did not allude to it in the paper otherwise than by the observation that the changes of temperature "being considered negligible were not investigated." Mr. Griffiths's remark that the variations of Clark cells depend on shifts of temperature does not appear to have much bearing on the variations which form the main subject of the paper, since the latter are, if the intrapolation of Ayrton and Cooper's results may be taken, about 15 times greater than the variations due to temperature. These variations are of the same order as the errors of observation shown in Ayrton and Cooper's diagrams. To account for these variations by differences of temperature between the cells, is to assume differences of from one-half to one degree between different cells. Under the conditions of the double box, lagging, and small temperature gradient observed, it is inconceivable that such differences could exist.

LI. *On the Distribution of Magnetic Induction in a Long Iron Bar.* By C. G. LAMB, *M.A.*, *B.Sc.** (Communicated by Prof. Ewing.)

THE following investigation was undertaken to determine the distribution of induction in a long cylindrical iron rod when it was subjected to various magnetizing forces. It was felt that the great variation of induction which must necessarily occur in a cylindrical bar would vitiate to some extent the ordinary assumptions made when employing rods for magnetic measurements.

In order to get a sufficiently clear notion of the induction distribution, and how it is produced, it is desirable to determine the following data:—

(P) A curve of magnetization of the bar as determined by means of a search-coil at its centre.

(Q) A series of curves at various fixed magnetizing forces showing the distribution of induction in each case.

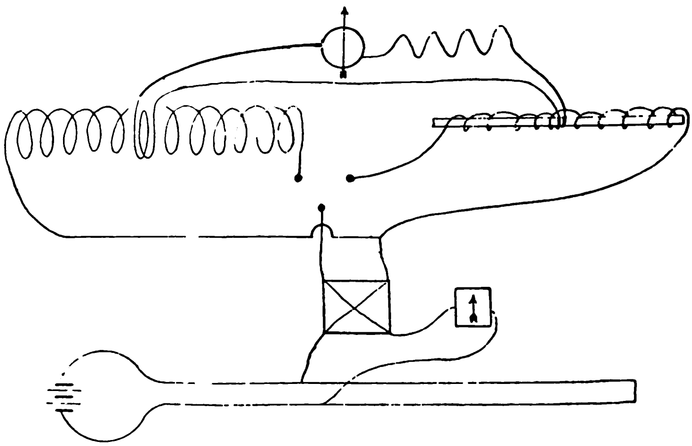
(R) A magnetization-curve of the bar when made into a ring, so that the induction is the same at every cross-section.

The bar used in the present case was a circular bar of Lowmoor iron; it was 0·485 cm. in mean diameter, and 123·4 cm. (48 inches) long, and before experiment was very carefully annealed. The magnetizing-coil was wound on a brass tube, somewhat longer than the specimen and of larger diameter; the specimen was fixed centrally inside this by ebonite rings near the centre and at one end. The search-coil was made as small as possible, and was wound on an ebonite bobbin attached to the end of another brass tube of such a size as just to slide over the rod, while the search-coil could just slide inside the tube covering the magnetizing-coil: on the inner tube marks were made showing when the search-coil was at the centre of the bar, and at other definite positions up to the end. For the determination of curve D it was kept at the centre. The larger magnetizing-currents were read on Weston standard ammeters calibrated by comparison with a

* Read June 9, 1899.

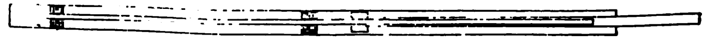
Crompton potentiometer; the small currents and those on the steep parts of the magnetization-curves were read directly by means of the potentiometer. The currents were adjusted by ordinary resistances for the larger values, or by a simple circular potential-slide as shown developed in fig. 1 for the smaller ones.

Fig. 1.



The ballistic galvanometer used was a Crompton "Midget" D'Arsonval, and was tested for proportionality by means of a standard field of known mutual-induction coefficient; it was found to be practically quite accurate. The standard field was also used throughout the experiment to calibrate the ballistic galvanometer, the connexions being shown in fig. 1. Fig. 2 shows the arrangement of the search-coil and specimen rod.

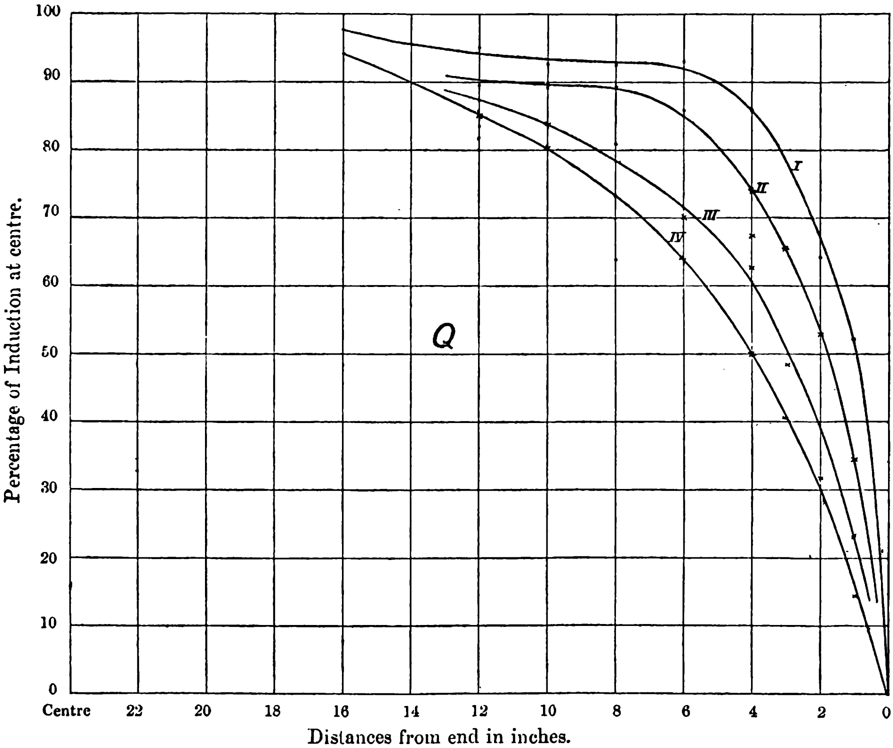
Fig. 2.



The results on reduction were carefully plotted to a large scale, and fig. 5 is a reduced copy of this curve; it connects the applied magnetizing force and the induction at the centre of the bar; we will refer to these as H_a and B_c.

The next step was to determine the curves of induction in the bar; these are called set Q (figs. 3 and 4), and nine were taken (numbered I. to IX.) at values of the magnetizing-current giving values of H_a shown in Table II. The points selected at which to measure the induction were: as near the end as possible, 1, 2, 3, 4, 6, 8, 10, 12, 14, 16, 18, 20 inches from that end, and at the centre (24 inches from end). A preliminary

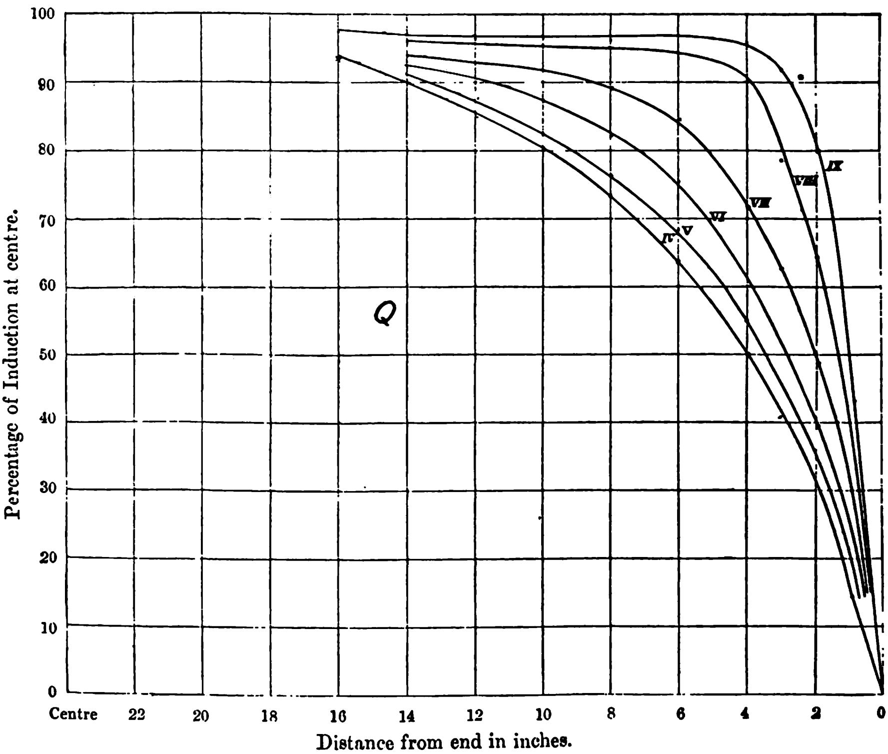
Fig. 3.



experiment showed that either half of the bar gave closely agreeing results at one definite value of H_a. Several observations were taken at each point, and from the mean of these the induction at each of the specified points was calculated as a percentage of that which occurred at the centre: these

values are given in Table I. Curves were carefully plotted to a large scale; those marked Q are given to show the mode of variation along the bar, the points near the centre are not shown as it will be noted from Table I. that some of the curves cross there owing to slight local differences in the iron; these points are omitted to avoid confusion. The curves subsequently referred to as "Curves Q" must be

Fig. 4.



taken as being drawn carefully through all the points, not those given here.

The bar was then very carefully welded and re-annealed, wound with a secondary coil and then a magnetizing-coil, and a reversal-curve again taken as for the bar. B, H curves were

TABLE I.

No. of Curve Q.	Inches from end.												
	Centre.	20.	18.	16.	14.	12.	10.	8.	6.	4.	3.	2.	1.
I. ...	100	......	99	96	94	95	93	93	93	86	78	64	52
II. ...	100	......	99	95	91	89·5	89·5	89·5	86	74	60	53	34
III. ...	100	......	100	97	92	87·5	83·5	78	70	37·5	48	37	23
IV. ...	100	......	99	94·1	90·5	85	80·2	73·3	63·7	50	40·5	31·8	14·3
V. ..	100	99·4	96·2	93	90·3	87·2	82·6	76·4	68	53·8	45	34	21·4
VI. ...	100	98·8	96·2	93·6	91·8	90·1	87·2	82·4	75	61	51·5	39·5	24·5
VII. ...	100	98·5	96	94	93·5	93	92	89	84·5	71·5	62	48·5	31
VIII. ...	100	99·0	99	96·5	96·2	96	95·5	95	94·5	92	78·5	64	41·5
IX. ...	100	99·8	98·5	97·5	97·5	96·5	96·5	96	97	96	92	80	43

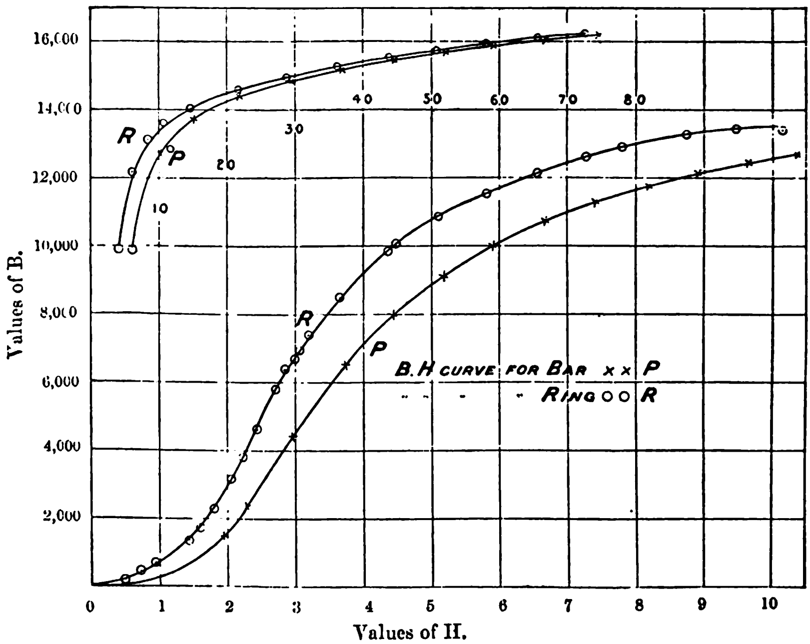
TABLE II.

Curve Q.	H_a.	H_m.	B_c.	B_m.	B_h.	$\frac{H_m}{H_a}$.	$\frac{B_m}{B_c}$.	$\frac{B_h}{B_m}$.	Equiv. length as fraction.	D in inches.
I. ...	0·74	......	162	146	... ..	......	......	......	·90	2·4
II. ...	1·49	1·18	572	477	468	·79	·84	·98	·835	4·0
III. ...	2·23	1·81	2,350	1,880	1,790	·81	·80	·96	·80	4·8
IV. ...	3·35	2·30	5,500	4,070	4,080	·69	·74	1·02	·74	6·3
V. ...	4·47	2·84	8,000	6,160	6,200	·64	·77	1·00	·77	5·4
VI. ...	6·70	3·84	10,700	8,680	8,800	·57	·81	1·01	·81	4·6
VII. ...	11·6	6·04	13,500	11,900	11,740	·52	·88	·99	·88	2·9
VIII. ...	20·0	11·50	14,300	12,900	13,600	·58	·90	1·04	·90	2·3
IX. ...	35·0	22·10	15,200	14,100	14,600	·63	·92	1·04	·92	1·9

drawn to suitable scales for high and low inductions so as to permit accurate reading of the quantities; the curve connecting B and μ was deduced and likewise plotted. Reduced copies of these curves are shown at R (figs. 5 and 6). References after must be taken to apply to the large-scale curves as before.

From curves Q we can easily deduce the mean induction by taking the area and base of the curves; this is given in Table II. under the heading B_m. The equivalent length of

Fig. 5.

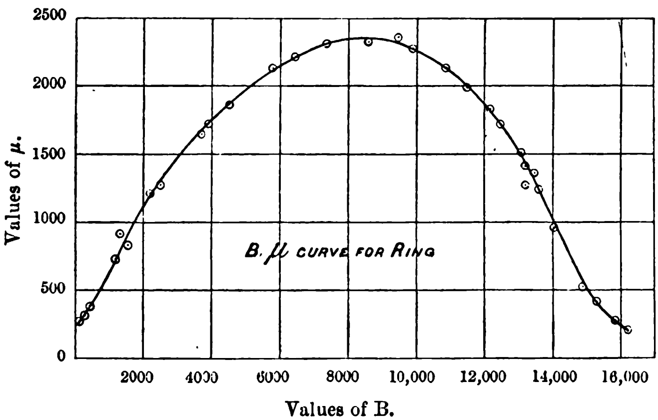


the bar will be the same proportion of its actual length that B_m is of B_c; this is also shown in the table. Half the difference between the true and the equivalent lengths will give the distance of the resultant pole from the end; this is tabulated under the heading D.

From Table I. and the curve P the actual inductions can be derived for the various points of the bar at any definite

value of H_a; if we can assume the annealing to have been good enough to permit us to take the bar as being the same under the two conditions (which is borne out by the close approximation of the curves towards the maximum induction), it is possible to deduce the distribution of H in the bar by cross reading from curve R the value of the H for each value of B at each point. Curves showing the distribution of H are given in set S (fig. 7). From these the value of the mean H was found as above described for B, and is tabulated under H_m in Table II. The induction this would have produced in the bar

Fig. 6.



was likewise read off curve R, and is given in Table II. under B_h; it is the induction that would have been produced by the mean value of H had the induction been uniform all along the bar.

The values of μ at successive points were likewise read off from curve, and these are shown in curves T (fig. 8).

Table II. contains also the ratios of some of the more important quantities.

On comparing P and R (fig. 5) it will be noticed that the differences in the abscissæ (giving the "demagnetizing force" at the centre) are far from proportional to B, until we get to regions where the induction through the bar is so high as to be

Fig. 7.

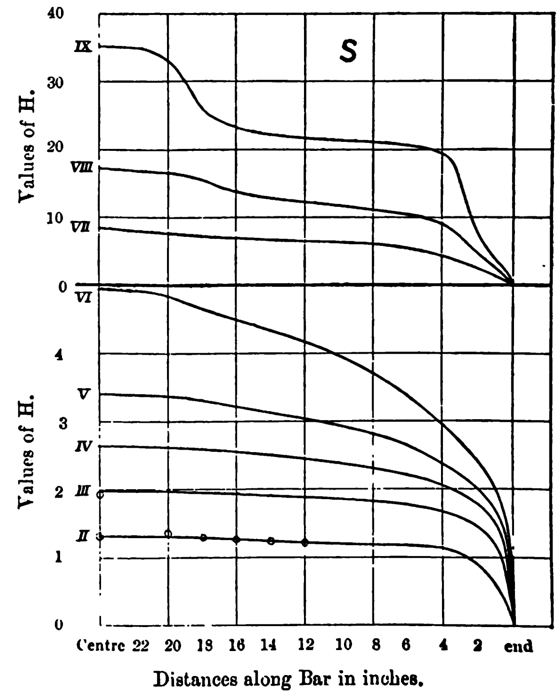
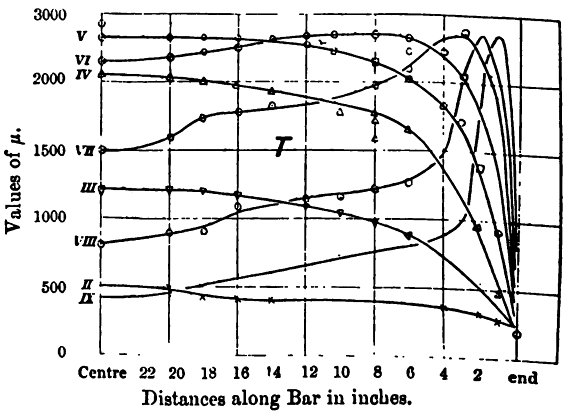

40
30
20
10
0
4
3
2
1
0
IX
VIII
VII
VI
V
IV
III
II
S
Values of H.
Values of H.
Centre 22 20 18 16 14 12 10 8 6 4 2 end
Distances along Bar in inches.

Fig. 8.


3000
2000
1500
1000
500
0
V
VI
IV
VII
III
VIII
II
IX
T
Values of μ.
Centre 22 20 18 16 14 12 10 8 6 4 2 end
Distances along Bar in inches.

practically constant all along (as shown in Q. IX., fig. 4); below such values the demagnetizing force is considerably *larger* than the amount that is usually taken for the length-ratio of 250 to 1 that was used here. Possibly some of this difference is due to imperfect annealing; but great care was taken, and curves P and R agree very well in those parts where from Q. IX. we see the induction is nearly constant along the bar. Furthermore, we should expect that with the great variations shown in set Q the leakage-paths would demand very varying proportions of the impressed magnetomotive force, instead of a constant portion, as is the usual assumption. In fact, we cannot possibly tell what value of B to take in the equation $H=\lambda B$ for demagnetizing force, since in the cylindrical bar (unlike the ellipsoidal) the leakage-flux is not independent of the induction. The ratio of H_m to H_a attains a minimum value near the point of greatest demagnetizing force, *i. e.* about 13,000; but stress cannot be laid on this owing to the reading-off from one curve to the other. It would seem, however, that the assumption of a constant demagnetizing factor is not quite satisfactory; consequently the magnetometric method with cylindrical rods, although extremely useful for comparative work, must be used with much caution in determinations of an absolute character.

On looking at curves Q we note that the induction drops most quickly somewhere in the neighbourhood of *maximum* permeability, that is, the induction leaks out more when the bar is the better conductor. This at first seems peculiar; but on referring to set T, one sees that in the cases where the permeability is increasing as we go along the bar, the induction keeps in the bar, and the quicker the permeability diminishes the more the induction leaks out. In fact, the leakage at any place depends on the average permeability of the part nearer the end of the bar, and not on the value at that particular place.

An interesting relationship is brought out in Table II. It will be noticed that the ratio of B_m and B_h ranges from about 2 per cent. less than unity to about 4 per cent. more. This is not a great variation from unity considering the number of operations to be gone through before arriving at the figures. It would thus appear that the induction in the iron and that

in the air so arrange themselves as to give a mean induction in the bar equal to that which the mean H would have produced had it been applied to the bar and no leakages taken place.

A very striking point is the great alteration in equivalent length of the bar; on the ellipsoidal assumption it is $\frac{2}{3}$ the length, and Kohlrausch's number for a cylinder is $\frac{5}{6}$. We see that, far from this being a constant number, it varies from ·9 to ·74; so that any determination depending on constancy of this quantity will be somewhat vitiated. A direct experiment was made to test the amount of displacement of the poles as follows:—A piece of iron wire was inserted in the magnetizing-solenoid used above, and a brass wire was attached so that it could be slid up or down, always inside the solenoid. It was first adjusted so as to be exactly opposite the needle of a magnetometer. Various magnetizing-currents were then sent round the solenoid, and the wire was moved along inside until the maximum deflexion was produced: the following numbers were obtained:—

Current.	Distance Wire was moved.
1·45	1·5 cm.
1·04	2·2
0·80	4·2
·57	6·5
·45	8·0
·38	12·0
·12	10·5

These results are of course quite rough, but the same general result is obtained as in the first experiments. The direct comparison of a ballistic and a magnetometric curve for the same bar would show that they approximately agree, since the factors causing variation of the equivalent length will affect both determinations in the same way; preliminary experiments have confirmed this. It is hoped to examine bars bent into incomplete rings and other forms.

The kind assistance of Mr. L. G. Walter, B.A., in the earlier experiments is gratefully acknowledged.

LII. *On the Magnetic Hysteresis of Cobalt.* *By* J. A. FLEMING, *M.A.*, *D.Sc.*, *F.R.S.*, *Professor of Electrical Engineering in University College, London*, A. W. ASHTON, *B.Sc.*, 1851 *Exhibition Scholar, and* H. J. TOMLINSON, *Salomons Scholar, University College, London* *.

[Plates II. & III.]

ALTHOUGH determinations have been made of the magnetic constants of cobalt by other observers, we have not been able to find any very complete set of observations on the magnetic hysteresis values of cobalt of known chemical composition corresponding to various cyclical magnetic forces of known range or maximum value. Having in our possession a cobalt ring of supposed fairly pure commercial cobalt, a series of magnetic experiments were undertaken with it, the results of which are embodied in the following tables and diagrams (see Plates II. & III.).

A rectangular sectioned circular ring of the metal was cast for us by Messrs. H. Wiggin & Co. of Birmingham, and turned up true in the lathe. The dimensions of the ring were then carefully measured and the mean values were found to be as follow :—

Mean outside diameter of ring	13·84 cms.
Mean inside diameter of ring	11·15 „
Diameter of mean perimeter of ring . .	12·50 „
Mean perimeter	39·26 „
Mean depth of cross-section of ring . . .	3·91 „
Mean cross-sectional area	5·23 sq. cm.
The mass of the ring was	1665 grams.
The volume of the ring	205·6 c. cm.
The mean density	8·1

The ring was then insulated with silk tape and wound over with four secondary coils of No. 36 S.W.G. silk-covered copper wire of 100, 50, 25, and 10 turns respectively, put on at quadrantal positions.

* Read June 23, 1899.

Over these secondary coils was wound a series of six primary coils of No. 18 S.W.G. cotton-covered copper wire laid on uniformly round the ring and having 195, 187, 180, 167, 161, and 155 turns respectively. The wire was well insulated with tape and shellac varnish. The ring so prepared was mounted on a board. From some of the turnings a chemical analysis of the metal was made for us by Mr. R. A. Hadfield, M.Inst.C.E., to whom we are greatly indebted.

The analysis showed that the metal had the following composition:—

Cobalt	95·95	per cent.
Nickel	·80	„
Iron	·91	„
Manganese . . .	·25	„
Silicon	·42	„
Carbon	1·36	„
Sulphur	trace.	
Phosphorus . . .	trace.	

The metal cannot therefore be considered as even approximately chemically pure. It contains nearly 1 per cent. of iron and 1 per cent. of nickel.

The ring thus prepared was submitted to a complete set of magnetic tests by aid of the ballistic galvanometer in the usual manner for the determination of the magnetic flux-density or induction corresponding to certain cyclic values of magnetizing force. It is unnecessary to give the details, excepting to state that the electrical measurements required were all ultimately referred to a standard resistance and Clark cell by means of a potentiometer. The ballistic galvanometer used was a Crompton-d'Arsonval with a movable coil having a periodic time of fourteen seconds.

The standardization of the galvanometer was effected in the usual manner with a standard secondary coil placed in the interior of a long primary solenoid of known dimensions. From the observations the magnetizing force (H) employed and the corresponding flux-density or induction (B) were

calculated and the results set out in the accompanying tables.

The first set of experiments consisted in taking a careful set of complete magnetic (B.H.) cycles (see fig. 1, Plate II.), and from these the hysteresis losses in ergs per cub. centim. per cycle corresponding to known maximum flux-densities were calculated by measuring the delineated areas. A Steinmetz curve was then constructed showing the relation of the hysteresis loss in each cycle to the maximum induction for that cycle (see fig. 2, Plate III.). This was converted by taking logarithms (ordinary) of both variables and plotting a logarithm curve (see fig. 3). This last curve proved to be nearly a straight line.

If W = magnetic hysteresis energy loss in ergs per cub. centim. per cycle, and
B = maximum flux-density (induction) during the cycle,
I = maximum magnetization during the cycle;

then the logarithmic curves show that over a wide range

$$W = 0{\cdot}01\ B^{1{\cdot}6}, \text{ or}$$
$$W = 0{\cdot}527\ I^{1{\cdot}62}.$$

These exponential expressions $W = \eta B^n$ and $W = \eta' I^{n'}$ for W in terms of B and I show that the relation found by Mr. C. P. Steinmetz* for iron and steels of various composition, and by Dr. A. E. Kennelly† for nickel, approximately holds good for this sample of cobalt, viz. that the hysteresis loss varies as the 1·6th power of the maximum flux-density during the cycle.

It is not a little curious that for materials differing so much as the above cast cobalt and soft annealed transformer iron the hysteretic exponent should in both cases be so near to 1·6. At low inductions the hysteretic exponent increases in value, however, in all cases. In the case of iron it is well known that magnetic hysteresis becomes zero before the maximum induction becomes zero. There is, in fact, a non-hysteretic range of cyclical magnetization. Hence it follows that for some

* 'Electrician,' vol. xxviii. pp. 384, 408, 425; also vol. xxxii. p. 672.
† 'Electrician,' vol. xxviii. p. 666 (1892).

small value of the maximum flux-density there must be a range of flux-density during which hysteresis increases from zero to a finite value; in other words, its rate of change is then very large. Hence $\log_{10} W$ will have a very large negative value corresponding to a certain small value of $\log_{10} B$; and accordingly it follows that the hysteretic constant *n*, which is represented by the tangent of the inclination of the line ($\log W$, $\log B$) cannot have the value 1·6 throughout all values of $B_{max.}$, but must increase as $B_{max.}$ decreases. That this is also the case for this cobalt sample is well shown by the lowest observation-points on the logarithm-curve (fig. 3). We conclude from the fact that the hysteretic exponent attains a higher value than 1·6 for very low inductions, that there is in cobalt also a non-hysteretic range of cyclical magnetization. It is usual to express W in terms of B; but inasmuch as W is really an expression for the behaviour of the metal *per se* corresponding to a given state of magnetization, it would be better to express W in terms of the magnetization I. Of the three magnetic vectors H, B, and I, I has reference to the properties of the material itself, B to those of the material and the space it occupies as well, and H may be looked upon as denoting the space qualities only.

Hence magnetic material qualities like W should be expressed in terms of I rather than B.

We have accordingly given the logarithmic curve connecting $\log_{10} W$ and $\log_{10} I$, I being the maximum magnetization during the cycle.

The fact that the hysteresis loss W during a cycle can be approximately expressed in terms of the maximum magnetization I during the cycle, seems to show that the work done in making a complete magnetic cycle, or carrying the magnetic molecules once completely round, is nearly a simple exponential function of the percentage of them which are collineated at the beginning and end of the cycle in the direction of maximum magnetization. The magnetization being the magnetic moment per unit volume, it follows that the work done in making a complete magnetic reversal of all the molecular magnets is a nearly simple exponential function

of the total resolved molecular magnetic moments in the direction of magnetization at the beginning and end of the operation.

It appears, therefore, that the work done in carrying the magnetic molecules in unit volume once round a complete cycle is nearly proportional to the 1·6th power of the aggregate magnetic moment of all the molecules estimated in the direction of the magnetic force.

The more the molecular magnets are collineated, that is, the greater the aggregate magnetic moment in a given direction, the greater the work required to effect a complete cyclic operation, the two magnitudes being related by a simple exponential relation. It would be interesting to determine if the mechanical work required to effect the complete magnetic reversal of a crowd of small compass-needles is proportional to the 1·6th power of the aggregate maximum magnetic moment before or after completion of the cycle.

The observations also furnished a simple magnetization-curve for the metal (fig. 4) and a series of values of the permeability stated in terms of the flux-density B and of the magnetic force H (fig. 5).

A study of these curves shows that this cast cobalt magnetically resembles many ordinary qualities of cast iron.

As a comparison, a series of similar observations were taken on a circular cast-iron ring.

This ring was prepared for us by Messrs. Easton, Anderson, and Goolden from a variety of cast iron used by them for dynamo purposes. The transverse section of the ring was rectangular.

The dimensions of the ring were as follows:—

Mean outside diameter . .	5·000 inch.
Mean inside diameter . . .	4·000 „
Depth of ring	0·500 „
Mean thickness	1·000 „
Mass of ring	1·8086 lbs.
Specific gravity	7·042

The ring was insulated and wound over with two primary coils of No. 20 S.W.G. cotton-covered copper wire, the inner layer having 248 turns and the outer 243 turns.

The primary coil was overlaid with two secondary coils of No. 36 silk-covered copper wire, one having 200 turns and the other 20 turns. The ring was submitted to the process above described for obtaining the cyclic magnetization-curves by determining with the ballistic galvanometer the value of B corresponding to various cyclic values of H. These observations were taken by Mr. H. I. Lewenz in January 1898, in the Pender Electrical Laboratory, University College, London.

The hysteresis loops were then carefully plotted out (fig. 6) by the same observer. From the measurements of area a Steinmetz curve was drawn (fig. 7), the ordinates of which represent ergs per cubic centimetre per cycle and the abscissæ maximum flux-density during the cycle. Also a curve (fig. 8) was drawn, the abscissæ and ordinates of which were the ordinary logarithms of the above quantities.

This latter curve proves to be a straight line over a considerable portion of its length. The inclination of the line, however, shows that the hysteretic exponent for this cast iron has a value of 1·97, considerably above the value obtained for the cast cobalt. It appears, therefore, as if the hysteretic exponent for cast iron at moderate and high inductions resembles that of wrought iron, and that of nickel and cobalt at very low induction densities. The permeability curves for this cast iron in terms of B and H as abscissæ are shown in fig. 9.

The chemical composition of the cast-iron ring we employed was kindly determined for us by Mr. R. A. Hadfield. His analysis showed the presence of 2 per cent. of silicon, 2·85 of carbon, and 0·5 of manganese. It was therefore not in any way abnormal. The following tables give an epitome of the observations on these cast cobalt and cast-iron rings.

Observations on the Magnetic Hysteresis of Cast Cobalt.

Cyclical Magnetization Curves.

Loop I.		Loop II.		Loop III.		Loop IV.	
H.	B.	H.	B.	H.	B.	H.	B.
6·67	911	13·23	2341	17·91	3106	25·76	4110
4·92	805	7·13	1928	9·17	2549	14·99	3196
3·24	676	3·75	1505	4·18	2018	7·28	2809
2·28	586	0	1092	0	1436	0	1807
0·99	453	− 3·21	418	− 3·42	727	− 5·28	420
0	347	− 5·36	− 286	− 5·92	− 205	− 8·91	−1005
−0·99	222	− 7·13	− 895	− 8·97	−1288	−12·46	−2023
−2·28	24	− 9·34	−1536	−12·89	−2312	−18·87	−3281
−3·25	−164	−11·72	−2069	−17·91	−3106	−25·76	−4110
−4·92	−537	−13·23	−2341				
−6·67	−911						

Loop V.		Loop VI.		Loop VII.		Loop VIII.	
H.	B.	H.	B.	H.	B.	H.	B.
30·53	4569	38·56	5216	48·54	5869	61·08	6519
17·64	3886	21·15	4397	26·79	4980	36·38	5705
7·75	3047	10·19	3536	10·91	3751	16·39	4440
0	2022	0	2177	0	2299	0	2418
− 5·23	494	− 6·23	201	− 5·60	674	− 6·11	550
− 9·51	−1182	−10·19	−1350	−10·91	−1376	−11·02	−1337
−14·41	−2490	−16·84	−2912	−17·03	−2846	−19·51	−3220
−20·85	−3535	−26·83	−4257	−28·20	−4383	−36·99	−5123
−30·53	−4569	−38·56	−5216	−48·54	−5869	−61·08	−6519

Loop IX.		Loop X.		Loop XI.	
H.	B.	H.	B.	H.	B.
75·46	7052	93·18	7622	114·03	8237
46·41	6224	52·56	6586	60 36	7020
18·43	4684	20·90	4940	24·33	5337
0	2460	0	2521	0	2626
− 6·74	354	− 7·17	240	− 7·86	− 27
−12·86	−1758	−14·97	−2273	− 16·71	−2571
−26·35	−4104	−33·73	−4825	− 37·84	−5167
−46·72	−5786	−57·64	−6366	− 71·10	−6989
−75·46	−7052	−93·18	−7622	−114·03	−8237

Magnetization Curve, Permeability, and Hysteresis Values of Cast Cobalt.

H.	B.	μ.	Hysteresis Loss in ergs per cc. per cycle.	Hysteresis Loss in watts per lb. per 100 cycles per sec.
6·67	911	137	452	·254
13·23	2341	177	2454	1·379
17·91	3106	173	3956	2·222
25·76	4110	160	6292	3·535
30·53	4569	150	7374	4·143
38·56	5216	135	8953	5·029
48·54	5869	121	10937	6·144
61·08	6519	107	13235	7·436
75·46	7052	93	14642	8·226
93·18	7622	82	16518	9·280
114·03	8237	72	18950	10·646

H is the maximum value of the magnetizing force during each cycle.
B is the maximum value of the flux-density or induction.
μ is the value of the permeability corresponding to the annexed values of B, and the hysteretic losses also corresponding to the values of H and B are given in ergs and watt-seconds.

Observations on the Magnetic Hysteresis of Cast Iron.

Magnetization Curve.

H.	B.	μ.	H.	B.	μ.	H.	B.	μ.
·19	27	139	8·84	4030	456	41·65	8071	181
·41	62	150	10·60	4491	424	56·57	8548	151
1·11	206	176	12·33	4884	396	71·98	9097	126
2·53	768	303	13·95	5276	378	88·99	9600	108
3·41	1251	367	15·61	5504	353	105·35	10036	95
4·45	1898	427	18·21	5829	320	120·60	10375	86
5·67	2589	456	26·37	6814	258	140·37	10725	76
7·16	3350	468	36·54	7580	207	152·73	10935	72

Observations on the Magnetic Hysteresis of Cast Iron (*cont.*).

Cyclical Magnetization Curves.

Loop I.

H.	B.	H.	B.	H.	B.	H.	B.
3·80	1475	·78	972	−1·13	240	−3·42	−1357
3·46	1440	·55	886	−1·62	− 103	−3·52	−1384
3·30	1426	·31	845	−2·09	− 494	−3·60	−1413
2 92	1377	0	750	−2·38	− 739	−3·80	−1475
2·82	1351	−·27	663	−2·60	− 906		
2·67	1335	−·30	627	−2·74	−1013		
2·11	1243	−·47	564	−2·86	−1074		
1·32	1090	−·89	374	−3·31	−1290		

Loop II.

H.	B.	H.	B.	H.	B.	H.	B.
5·53	2545	·59	1762	−1·49	730	−5·09	−2348
4·65	2473	·37	1677	−2·33	− 648	−5·19	−2396
3·81	2392	0	1554	−3·39	−1251	−5·53	−2545
3 51	2355	−·27	1421	−3·63	−1469		
2·80	2247	−·37	1388	−3·83	−1642		
2·32	2160	−·59	1285	−4·70	−2150		
1·47	2000	−·85	1133	−4·95	−2286		

Loop III.

H.	B.	H.	B.	H.	B.	H.	B.
8·75	3865	2·75	3228	− ·61	2194	−7·32	−3436
7·46	3781	·89	2799	−1·62	1583	−7·99	−3626
5·19	3569	·38	2605	−2·73	210	−8·75	−3865
4 67	3484	0	2465	−4·15	−1482		
4·17	3441	−·38	2301	−4·97	−2348		

Loop IV.

H.	B.	H.	B.	H.	B.	H.	B.
19·14	5972	1·65	4239	− ·48	3536	− 7·51	−3128
14·77	5825	·89	4047	− ·94	3316	−14·45	−5188
11·38	5646	·47	3950	−1·80	2722	−15·94	−5475
10·92	5526	0	3804	−4·48	− 735	−19·14	−5972
10·21	5496	−·31	3663	−6·36	−2473		

Cyclical Magnetization Curves (*continued*).

Loop V.							
H.	B.	H.	B.	H.	B.	H.	B.
71·50	8930	6·31	5760	−10·52	−4189	−43·47	−7996
43·86	8262	0	4290	−24·10	−6628	−49·00	−8116
32·07	7715	−2·20	2937	−31·75	−7289	−51·11	−8239
10·55	6311	−7·04	−2473	−36·58	−7640	−71·50	−8930

Loop.	B (max.).	Hysteresis Loss.	
		Ergs per cc. per cycle.	Watts per lb. per 100 cycles per sec.
I.	1475	466	·300
II.	2545	1288	·829
III.	3865	2997	1·934
IV.	5972	7397	4·765
V.	8930	13423	8·658

The result of experiments with the cast cobalt ring is to show that although in general form its magnetization-curve resembles that of cast iron, its hysteretic exponent is similar to that of annealed soft iron. The absolute hysteresis values corresponding to various maximum flux-densities are, however, considerably larger than those of a typical variety of cast iron.

Discussion.

Prof. Everett referred to the fact that the sample of cobalt contained about 1 per cent. of iron, and said that it would be interesting to know how cobalt free from iron would behave.

Mr. Blakesley said that the hysteresis curves obtained from the step by step method could not be applied to dynamos, because the time taken to perform the cycle altered the shape of the curve. He would like to see the curves for cobalt determined in cases where the cycles were quickly executed.

LIII. *Demonstration of Prof. T. W. Richards's Method of Standardizing Thermometers.* By R. A. LEHFELDT, *D.Sc.**

[Abstract.]

THIS method depends upon the ordinary latent-heat principle for maintaining constant temperature, but it includes the consideration of the fusion of systems with more than one component. If there are c components and p phases, then the number of degrees of freedom of the system is $[c+2-p]$. When this is zero, the temperature and the pressure of the system are perfectly definite. Thus the following formula represents the four phases to be equilibrated in the case of Glauber's salt:—

$$\begin{cases} Na_2SO_4\,10H_2O \\ Na_2SO_4 \\ \text{Solution} \\ \text{Vapour.} \end{cases}$$

The transition point for this system is 32°·379 on the hydrogen scale, and can be determined to within a few thousandths of a degree.

Prof. Richards † has determined the temperature of equilibrium in several useful cases. The salts are put into a test-tube in an air-bath formed between it and a second test-tube; the whole is then heated in a beaker of water over a small flame, or, preferably, in a thermostat kept about ½ degree above the transition point. If the salt is pure, and care is taken to avoid the effect of supersaturation, this method is highly satisfactory. It gives an extensive range of fixed-points, and is especially useful in thermometry for fixed-points between 0° and 100°. A few of these temperatures may be noted:—

Sodium chromate	19°·63 C.
Sodium carbonate	35·2
Sodium thiosulphate . . .	47·9
Sodium bromide	50·7
Manganese chloride . . .	57·7
Trisodium phosphate . . .	73·3
Barium hydroxide	77·9

* Read April 21, 1899.

† *Zeits. f. phys. Chem.* xxvi. p. 690 (1898), xxviii. p. 313 (1899).

Discussion.

Mr. J. A. Harker asked how long the temperature remained constant. Richards's method would appear to be a useful one for standardizing open scale-thermometers, as it would obviate the necessity for the provision of auxiliary bulbs and the 100°-point. It might also be of service as a fixed-point for meteorological thermometers.

Mr. Blakesley said that sodium chromate was represented by a very useful temperature. Could this substance be regarded as sufficiently stable to give a satisfactory fixed-point?

Dr. Lehfeldt, in reply, said that the time taken by the transformation was certainly several hours. All the fixed-points mentioned were theoretically as definite as that corresponding to sodium sulphate, but they had not been so accurately determined.

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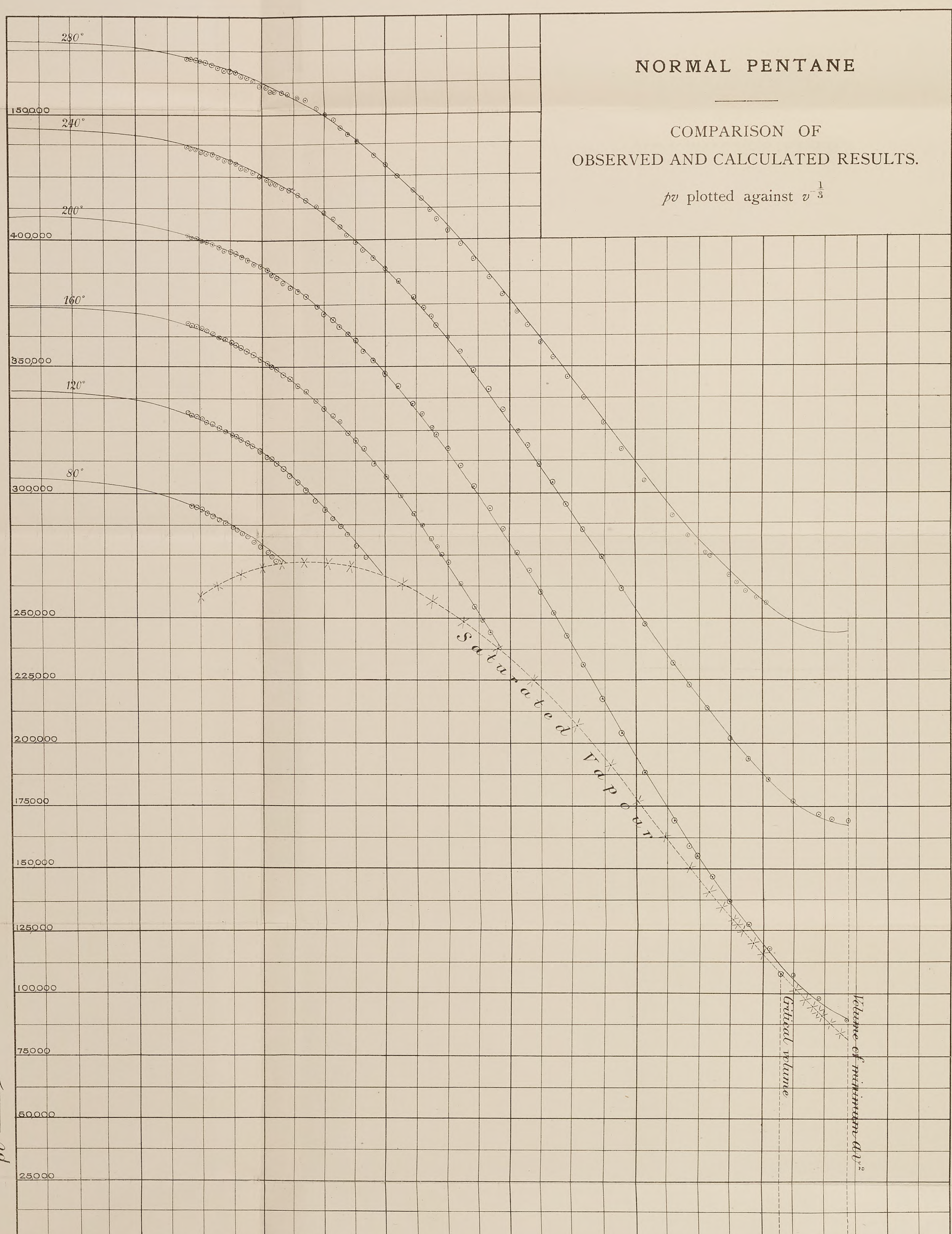

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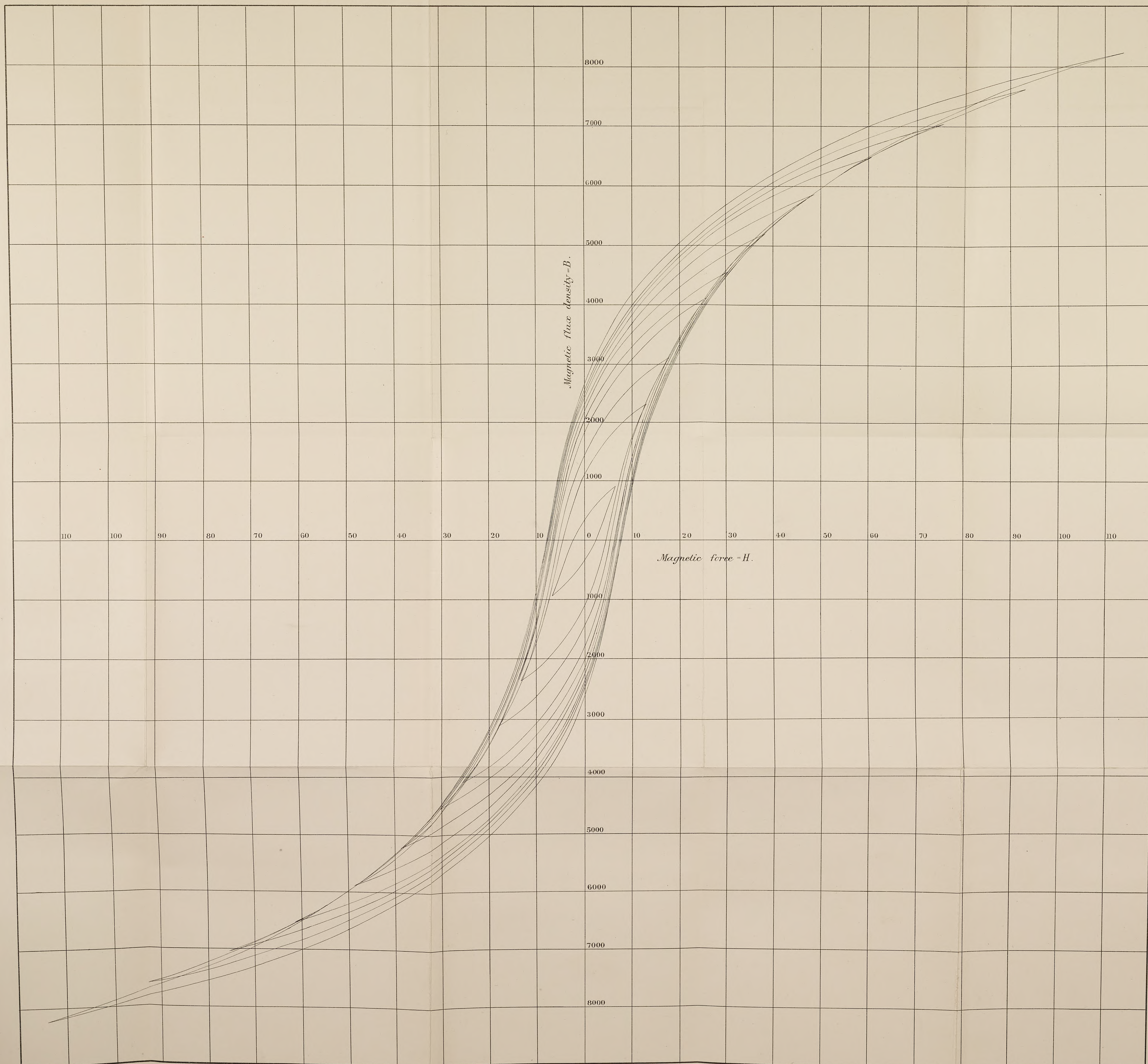

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SERIES OF HYSTERESIS LOOPS FOR COBALT RING

Fig. 1.

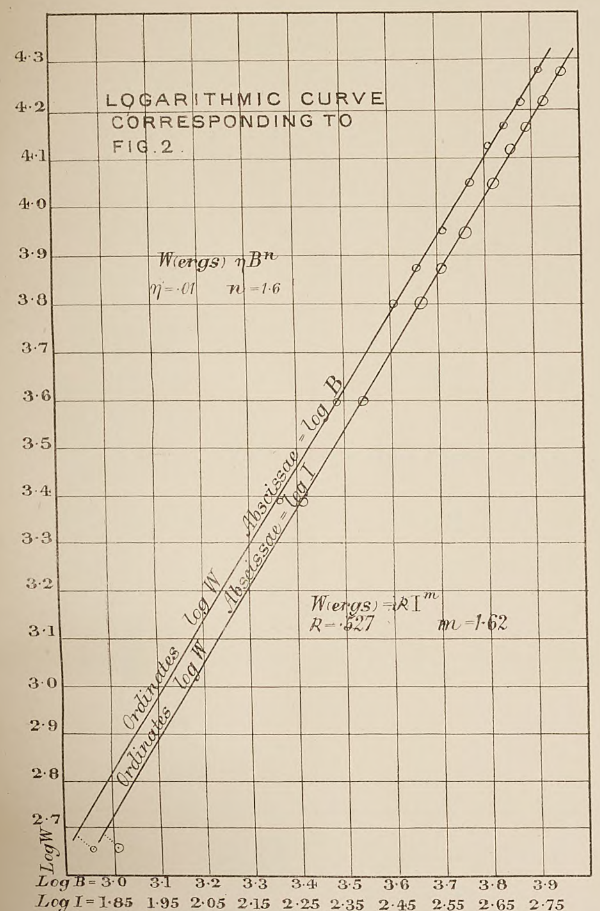
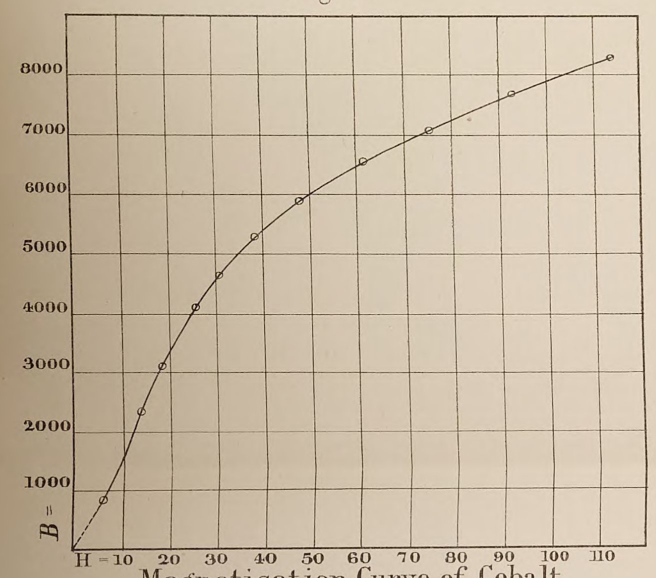


Fig. 3.



Magnetisation Curve of Cobalt
Fig. 4.

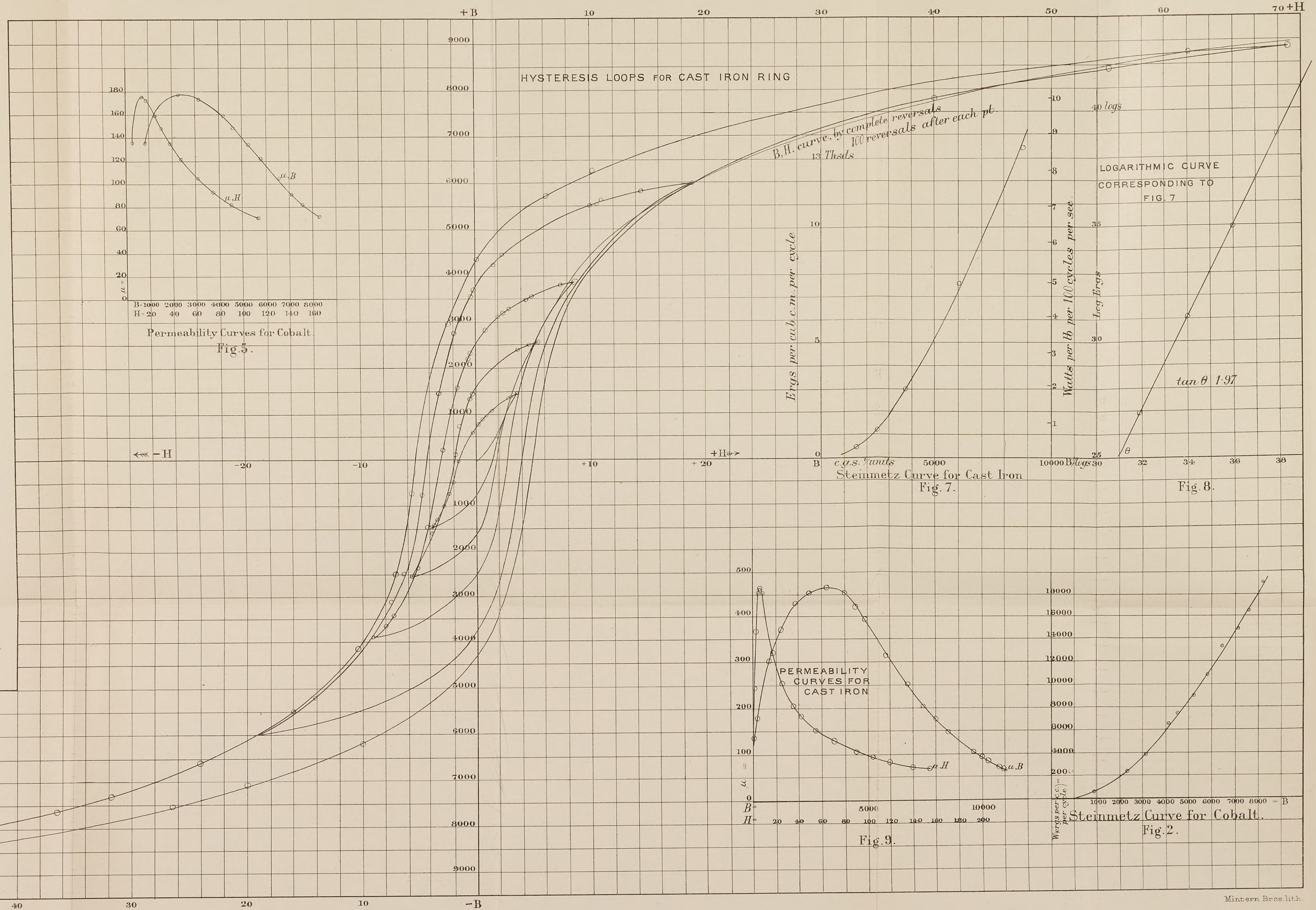


Fig. 6.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1897-98.

February 26th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following was elected a Fellow of the Society:—

Prof. K. PEARSON, F.R.S.

The following communications were made:—

"On the Photography of Ripples." By Mr. J. H. VINCENT.

"On the Thermoelectric Properties of some Liquid Metals." By Mr. W. B. BURNIE.

March 12th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following was elected a Fellow of the Society:—

Dr. J. H. HARKER.

The following communication was read:—

"On a Mechanical Cause of Homogeneity of Structure and Symmetry Geometrically Investigated; with Special Application to Crystals and to Chemical Combination." By Mr. WILLIAM BARLOW.

March 26th, 1897.

Meeting held in the Physical Laboratory of the Finsbury Technical College by invitation of Prof. S. P. Thompson.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—
Dr. Dawson Turner and Mr. James Quick.

Mr. R. Appleyard exhibited some experiments on Liquid Coherers and Mobile Conductors.

Prof. Dalby exhibited the following pieces of apparatus:—

1. Ewing's Reading Telescope.
2. A Kinematic Slide.
3. An Inertia Apparatus with trifilar suspension.
4. A Kinematic Hook-Gauge.
5. A Wilberforce Spring.

Prof. S. P. Thompson exhibited the following pieces of apparatus:—

1. A Model illustrating the Synthesis of two Opposite Circular Motions of equal Period and Amplitude.
2. A Model illustrating the Synthesis of two Simple Harmonic Motions of equal Period and Amplitude with any difference of Phase to form Circular Motion.
3. The Projection of Arago's Rotations.
4. Some experiments with Heat-indicating Paint.
5. A new Model illustrating the Propagation of a Hertz Wave.

April 9th, 1897.

Shelford Bidwell, Esq., President, in the Chair.

Mr. Garrett and Mr. Lucas communicated a paper "On a Nickel Stress Telephone."

Mr. W. A. Price communicated a paper "On Alternating Currents in Concentric Conductors."

Mr. W. B. Morton communicated a paper "On the Effect of Capacity on Stationary Electric Waves in Wires."

May 14th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following was elected a Fellow of the Society:—

Mr. F. R. FLEW.

Mr. WATSON exhibited an Instrument for the Comparison of Thermometers.

Dr. MORRIS communicated a paper "On the Effect of Temperature on the Magnetic and Electrical Properties of Iron."

Mr. APPLEYARD read a paper "On the Formation of Mercury Films by an Electrical Process."

May 28th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following was elected a Fellow of the Society:—

Mr. G. THOMAS-DAVIES.

Dr. A. A. GRAY communicated a paper "On the Perception of the Difference of Phase by the two Ears."

Mr. J. ROSE-INNES read a paper "On the Isothermals of Isopentane."

June 11th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. R. H. HOUSMAN and F. W. LANCHESTER.

Mr. C. S. WHITEHEAD read a paper "On the Effect of Sea-Water on Induction Telegraphy."

Mr. BLAKESLEY read a paper "On a New Definition of Focal Length, and an Instrument for its Determination."

Drs. J. E. MYERS and F. BRAUN communicated a paper "On the Decomposition of Silver Salts under Pressure."

Prof. J. A. FLEMING read a paper "On a new way of Determining Hysteresis in Straight Iron Strips."

June 25th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

Mr. SUTHERLAND communicated a paper "On a New Theory of the Earth's Magnetism."

Dr. KUENEN read a paper "On Experiments in Critical Phenomena."

Dr. BARTON communicated a paper "On the Attenuation of Electric Waves in Wires."

Mr. SEARLE read a paper "On the Steady Motion of an Electrified Ellipsoid."

October 29th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

Prof. STROUD described the Barr and Stroud Naval Range-Finder, and a Telometrical Focometer and Spherometer.

Mr. A. S. E. ACKERMANN exhibited some Experiments illustrating Surface-Tension.

November 12th, 1897.

Dr. G. JOHNSTONE STONEY, Vice-President, in the Chair.

Mr. J. ROSE-INNES read a paper "On the Isothermals of Ether."

Messrs. F. S. SPIERS, F. TWYMAN, and W. L. WATERS communicated a paper, "On the Variations of the Electromotive Force of the H-form of Clark Cells with Temperature."

November 26th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following was elected a Fellow of the Society:—

Mr. S. W. J. SMITH.

Mr. R. APPLEYARD read a paper "On the Failure of German Silver and Platinoid Wires."

Prof. PERRY read a note on Thermodynamics.

December 10th, 1897.

SHELFORD BIDWELL, Esq., President, in the Chair.

Mr. A. CAMPBELL showed two simple Lecture Experiments and an Apparatus for the Self-acting Temperature Compensation of Standard Cells.

Mr. J. ROSE-INNES read a paper "On Lord Kelvin's Absolute Method of Graduating a Thermometer."

January 21st, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. C. O. BARTRUM, and W. FRANCIS, Jun.

Prof. FITZGERALD exhibited some Photographs showing the Zeeman Effect which had been taken by Mr. Preston.

Prof. LODGE opened a discussion on Electrical Signalling without Connecting Wires.

Prof. S. P. THOMPSON exhibited a new form of Tesla Oscillator.

Annual General Meeting.

February 11th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. G. S. ECKFORD and Prof. A. LODGE.

The following Report of the Council was read by the Secretary:—

During the past Session the work of the Physical Society has been carried on steadily and satisfactorily.

Thirteen meetings have been held at which papers have been read. One of these, on March 26th, 1897, was held at the Finsbury Technical College, Leonard Street, City Road, by the kind invitation of the Foreign Secretary, Prof. S. P. Thompson. There was a large attendance of Fellows and Visitors, and great interest was felt in

Prof. Thompson's laboratory. All the other meetings were held in the rooms of the Chemical Society at Burlington House.

Eleven new Fellows have been elected into the Society, and the Council regrets to report that among those who have been lost to the Society through death are Lord Sackville Cecil, the Rev. H. Freeman, Mr. Adam Hilger, and Mr. J. W. Rodger. Short memoirs of these will be published in the 'Proceedings.' There have been two resignations, so that the numbers of the Society are more than kept up.

The only other matter that calls for special notice is the completion of the negotiations with the Institution of Electrical Engineers. The result of this will be that the combined strength of the two Societies will be devoted to the perfection of the system of Abstracts of Physical Papers already carried on by this Society for the past three years.

Lord Sackville Arthur Cecil was the son of the second Marquess of Salisbury by his second wife, now the Dowager Countess of Derby, and was born in 1848.

His education was directed by his father, who in 1858-9 was Lord President of the Council, and thus had charge of what is now the Education Department. The late Lord Salisbury much studied the subject of education, and had views of his own as to the manner in which boys should be brought up. As a consequence, Lord Sackville was accustomed to use his hands from his childhood, and the habit of observation thus acquired was most valuable to him in afterlife. He was very methodical and acted up to the maxim: "A place for everything, and everything in its place." He was a pupil at Wellington College for a short time in the early days of the school, and there his study was a type of neatness; he had a cabinet of tools, and if any of them were not in their proper places he seemed quite distressed. This methodical habit was continued through life, and those who knew the study in his house at Hayes Common will remember the perfect order of its arrangement.

After leaving Wellington College, Lord Sackville was placed under the care of a private tutor, and about this time he made a practical acquaintance with the working of the electric telegraph

in the signal-box at Hatfield Station, and he obtained a pair of single-needle instruments, by which he connected two distant rooms at Hatfield House. Later, he and his two younger brothers went to Paris with his tutor for a year, and there, for the first time, he saw an electric bell, which so much pleased him that he introduced them into the house on his return home. At the early age of 14 he devised an ingenious mode of connecting by electricity the turret clock at Hatfield with a clock in the house, which was driven by the current. He afterwards spent more than four years with the same private tutor at the old Manor House of Cranborne; here he was prepared for entering the University of Cambridge; he matriculated at Trinity College as a Fellow Commoner in 1865, and took his degree in Applied Science in 1869.

While still an undergraduate Lord Sackville Cecil was appointed an Associate Juror for this country at the Paris Exhibition in 1867 for Telegraphic Apparatus and Processes, Sir Charles Wheatstone being the Juror for that Class.

After leaving Cambridge, Lord Sackville worked in the Great Eastern Railway workshops at Stratford, and also in those of the Great Northern Railway at Doncaster. For a short time he was attached to the engineers' staff of the French Atlantic Cable, and subsequently acted as chief electrician in the laying of the Marseilles-Bona Cable.

From 1878 to 1880 he was Assistant General Manager of the Great Eastern Railway, and General Manager of the Metropolitan District Railway from 1880 to 1885.

After retiring from the latter post, which he found very arduous, Lord Sackville Cecil again devoted his attention to electrical matters, and became a Director of the Eastern Telegraph and of the Brazilian Telegraph Companies. He died on January 29, 1898.

He was an enthusiast for work and was most conscientious in the performance of all his duties, and no consideration of personal comfort, or even the opinion of others, would deter him from doing what he believed to be right. He made himself acquainted with all the details of the work he had to direct, so that in the event of anything going wrong he should be able at once to properly correct it. Although it does not appear that he published any original investigations, yet this was probably more due to an innate diffidence than to any other cause. He always showed an intense interest in all scientific subjects, and many would be surprised to

hear of the numerous applications of physical and chemical science which he made at different times as occasions served.

He was one of those men whose true value we begin to learn after they are taken from us.

Lord Sackville Cecil was one of the original members of the Physical Society.

The Rev. ALEXANDER FREEMAN was born January 28th, 1838, and was educated at the Merchant Taylors' School, whence he proceeded to St. John's College, Cambridge, in 1857, having obtained a Scholarship on the Foundation. He graduated a Fifth Wrangler in 1861, and in February 1862 he gained the Chancellor's Gold Medal for Legal studies. He was elected a Fellow of his College in May 1862, and took his M.A. degree in 1864. He acted as deputy for the Plumian Professor of Astronomy at Cambridge from 1880 to 1882. He translated and compiled an English edition of Fourier's *Analytical Theory of Heat*, and added to it many valuable original notes, and recently brought out a new edition of Cheyne's *Planetary Theory*. To the *Monthly Notices* of the Royal Astronomical Society, of which he was elected a Fellow in 1864, he contributed about a dozen papers. He further acted as Director of the Saturn Section of the British Astronomical Association, from which office he retired, on account of failing health, in 1895. His death took place somewhat suddenly on Saturday, June 12, 1897. He was admitted a Member of the Physical Society during the first session. In addition he was a Fellow of the Cambridge Philosophical Society, and a Member of the London Mathematical Society.*

By the death at the early age of 29 of JAMES WYLLIE RODGER the Society has lost one of its most promising Fellows. Although he was a comparatively newly-elected Fellow, having joined the Society in 1895, he always took part in the discussions on any papers of a physico-chemical nature read before the Society, and so was known to most of the Fellows.

He was born at Stewarton, N.B., on December 11, 1867, and was educated at Kilmarnock Academy, and at the Royal College of Science, South Kensington. His career as a student was interrupted by illness, but nevertheless he won all the chief prizes and

* This note is an abstract of one appearing in the *Proceedings* of the London Mathematical Society.

besides took a very prominent place in the management of the College societies. In 1889 he was appointed an Assistant in the research laboratory of the Chemical division of the Royal College of Science, and during the next few years he was the joint author with Dr. Thorpe of a number of papers bearing on Chemical Physics. The most important of these were on the Viscosity of Liquids, and were published in the Transactions of the Royal Society. One of these papers was selected for the Bakerian lecture in 1894. The lecture was delivered by Rodger, and all who were present will have a lively recollection of the very admirable manner in which it was delivered. Although he was the youngest Bakerian Lecturer, there can be no doubt that the lecture had rarely, if ever, been better delivered. During the evenings of the years 1893-5, while his teaching work occupied his whole time during the day, Rodger was spending his evenings working with his friend Mr. W. Watson on the magnetic rotation of the plane of polarization of light in liquids, the results of the first part of the work being published in the Transactions of the Royal Society. It is to be feared that working as he did at this time, often till late into the night, was in a certain measure responsible for his subsequent breakdown.

In addition to his experimental work Rodger was a frequent contributor to *Nature* and *Science Progress*, his series of papers on the "New Theory of Solutions" in the latter being particularly noticeable.

Of singularly attractive appearance and manners, popular with his fellows, a good teacher, and a first-rate lecturer, he had done enough solid work to prove that, if life and health had been spared, he would have won an honourable place in the ranks of English science. This was, however, not to be, for early in 1896 he was attacked by influenza, and from this he never recovered. He lingered on for about a year and then died, on May 5, 1897.

The Report of the Council was adopted.

The following were elected Honorary Fellows of the Society:—

Prof. RICARDO FELICI and Prof. EMILIO VILLARI.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—SHELFORD BIDWELL, M.A., LL.B., F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. GLADSTONE, F.R.S. ; Prof. G. C. FOSTER, F.R.S. ; Prof. W. G. ADAMS, M.A., F.R.S. ; Lord KELVIN, D.C.L., LL.D., F.R.S. ; Prof. R. B. CLIFTON, M.A., F.R.S. ; Prof. A. W. REINOLD, M.A., F.R.S. : Prof. W. E. AYRTON, F.R.S. ; Prof. G. F. FITZGERALD, M.A., F.R.S. ; Prof. A. W. RÜCKER, M.A., D.Sc., Sec.R.S. ; Capt. W. DE W. ABNEY, R.E., C.B., D.C.L., F.R.S.

Vice-Presidents.—C. VERNON BOYS, F.R.S. ; Major-General E. R. FESTING, R.E., F.R.S. ; G. GRIFFITH, M.A. ; Prof. J. PERRY, D.Sc., F.R.S.

Secretary.— H. M. ELDER, M.A.

Foreign Secretary.—Prof. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.—Dr. E. ATKINSON.

Librarian.—W. WATSON, B.Sc.

Other Members of Council.—Prof. H. E. ARMSTRONG, D.Sc., F.R.S. ; WALTER BAILY, M.A. ; L. CLARK, C.E., F.R.S. ; A. H. FISON, D.Sc. ; R. T. GLAZEBROOK, M.A., F.R.S. ; Prof. A. GRAY, LL.D., F.R.S. ; Prof. J. VIRIAMU JONES, M.A., F.R.S. ; S. LUPTON, M.A. ; Prof. G. M. MINCHIN, M.A., F.R.S. ; J. WALKER, M.A.

Votes of thanks were passed to the Auditors, the Officers and Council, and to the Chemical Society.

The President then gave the following Address :—

Address by the President,

SHELFORD BIDWELL, M.A. LL.B. F.R.S.

It has been thought desirable by the Council that the proceedings at the Annual General Meeting of the Physical Society should include an Address by the President.

The form assumed by the presidential address in the case of other scientific bodies varies greatly. In the absence of any definite precedent, I propose on this occasion to begin with a few words upon the history and objects of our Society, and afterwards to glance briefly at the principal events of physical interest which have occurred during the past year.

The Physical Society was founded in 1874, and owed its origin mainly to the initiative of the late Prof. Guthrie. From the first years of its existence up to the present time it has included among its members nearly all the leading physicists of the United Kingdom.

The principal objects of the Society, as stated in the Memorandum of Association, are to receive and discuss written or oral communications relating to Physics: to invite the exhibition of apparatus for physical research and of experiments illustrative of physical laws and phenomena, and to print and publish any communications made to the Society and other matter relating to Physics.

In the early days our meetings were, by permission of the Lords of the Committee of Council on Education, held in the Physics lecture-room of the Royal College of Science, at 3 o'clock on Saturday afternoons, members being allowed the free use of the laboratory apparatus for the illustration of their papers. The proceedings were at that time rather less formal than is customary at present. The papers communicated were rarely if ever "referred" before being read; often, indeed, they were read long before they were actually written, while a large proportion of the communications were of a purely oral character and never intended for publication at all, except perhaps in the short notes which the reporter sent to the scientific journals. Special prominence was given to experimental demonstrations illustrative not only of original researches but also of such work, carried on outside the Society, as happened to be attracting attention at the time; and many little devices were exhibited which could be employed for purposes of lecture illustration or class instruction.

Under the somewhat lax *régime* which then prevailed it necessarily happened that the communications made to the Society were not always of a very high order of merit, and, in fact, they were sometimes blemished by serious errors. But the demolition of the authors added to the interest and liveliness of the discussions, and no one was much the worse. For from the very beginning the Council has always been careful not to print in the 'Proceedings' anything that was not of sound scientific value, and while the number of important papers that have been published through the medium of the Physical Society is large, very little of doubtful quality has had a place in the Journal.

The first material change in our routine took place in the year 1889, when the day and hour of meeting were altered from Saturday afternoon at 3 to Friday at 5 o'clock. This was done primarily with the object of meeting the convenience of the younger members of the Society, who, it was thought, would show a greater devotion to science if they were not called upon to sacrifice their Saturday half-holidays to its claims.

I may here mention that the Council has more than once seriously considered whether it might not be expedient to hold our meetings in the evening. Many no doubt would consider this preferable, but the balance of convenience appears to be clearly in favour of the afternoon; and it is worthy of remark that at the Royal Society, as recently testified by the President, the attendance has largely increased since the meetings have been held at half-past four instead of eight o'clock.

While domiciled at South Kensington the Physical Society published and presented to its members a number of valuable books, including, among others, Prof. Everett's well-known treatise on the C.G.S. System of Units and the works of Wheatstone and of Joule. It also issued twelve volumes of 'Proceedings,' in which were collected such of the communications to the Society as had been approved for publication. By an arrangement with the proprietors of the 'Philosophical Magazine,' the same papers were also (as now) printed in that journal, being thus, to the authors' great advantage, assured of a wide circulation throughout the scientific world.

In the same period, notwithstanding the small amount of the annual subscription paid by members and of the composition fee for life membership, the Society's income so far exceeded its expenditure that it was able to accumulate and invest a capital of nearly £3000.

When the Society entered upon its twenty-first year with a position which, if somewhat unpretending, was well recognized and firmly established, it was felt that the time had come when in the interests of physical science something more than had been already achieved might fairly be demanded of it. British physicists had long been at a serious disadvantage, in that they were without any means of readily ascertaining what was being done by their fellow-workers in other countries; with the multiplication of scientific literature the need of some periodical digest similar in character to the German *Beiblätter* was becoming year by year more urgent. To endeavour to meet this want was a duty which clearly devolved upon the Physical Society, and the Council anxiously considered the question whether the publication of monthly abstracts of physical papers appearing in foreign journals could be undertaken by ourselves.

The only serious objection to the enterprise was of a financial nature. The work, if it were to be carried out efficiently, would certainly necessitate an annual expenditure exceeding by some hundreds of pounds the total income of the Society. This could only be met by raising the amount of the annual subscription and composition fee, which, as I have mentioned, were unusually low. But it is a delicate and difficult matter to ask existing members of a society for increased subscriptions unless very excellent reasons can be shown for the demand. The Council therefore determined that they would publish a series of Abstracts for one year at least before taking any steps to provide additional income, defraying the cost from cash in hand, and, if need should arise, drawing upon the invested capital. In this way it was hoped to convince members of the utility of the undertaking which they were to be called upon to support.

I need not remind you of the highly satisfactory result of the experiment. The work of the able and assiduous body of abstractors, whose names appear on the cover of our 'Proceedings,' was on all sides cordially approved; and at a Special General Meeting, held in 1896, a resolution submitted to the Society for increasing the subscription to £2 2*s.* per annum was passed almost, if not quite, unanimously. The number of those who in consequence of this increase have resigned their membership has been unexpectedly small, while on the other hand many of the life-members have, in response to the invitation issued to them, voluntarily contributed an additional £15 15*s.* to the funds of the Society, in recognition

of the fact that they are now in enjoyment of greater and more costly advantages than were contemplated at the time when they paid their very moderate composition fees. To such as have not yet responded, I venture to repeat the invitation.

Although the Abstracts were actually published for two years before the increased subscriptions began to come in, the whole cost was met out of uninvested cash supplemented by grants liberally made by the British Association and the Royal Society, and it was never found necessary to draw anything at all from the reserve fund. I wish to emphasize this fact because the abundant caution properly exercised by the Council in entering upon a new and uncertain enterprise appears to have led to a very general impression that the Society had outrun its means and was on the verge of bankruptcy, whereas, in truth, it was never in a more prosperous financial condition than it is at present.

On October 26th, 1894, the Society met for the first time in this room. Although the privileges so generously accorded to us by the authorities at South Kensington were highly valued, it was nevertheless deemed advisable that we should leave the home of our youth and seek a footing in Burlington House, the headquarters of scientific associations. Here the Chemical Society offered us a most kindly and cordial welcome and provided us with a meeting place which is not only more generally accessible than the old one, but is also on other grounds more convenient to most of our Fellows.

By this time the general affairs of the Society had assumed a more business-like condition. Amongst other things greater care was exercised with regard to the acceptance of communications. I need hardly say that no paper is in these days allowed to be read unless it has been first referred to some competent authority and favourably reported upon. The most distinguished physicists in the kingdom have given their services as referees, and our heartiest thanks are due to them for the care and patience which they have ungrudgingly bestowed upon a somewhat ungrateful task.

The practice was also introduced of putting the more important papers into type before they were read and distributing the proofs among such of the Fellows as were known to be specially interested in the subjects to which they related. This course has led to a considerable improvement in the value of the discussions.

The most recent step in advance consists in the adoption of a scheme for greatly extending the list of journals from which the monthly abstracts are made. With regard to this scheme, which

was finally approved by the Council at the meeting three weeks ago, I shall say a few words presently.

In reviewing the history of the Society for the past 24 years, there occurs to me only one material point in respect of which the old times were better than the new. It must be confessed that there has been a lamentable falling off from the abundance of experimental illustration, which in former days added so greatly to the attractiveness of our meetings. The remedy lies chiefly in the hands of our Fellows. It is true we have not the same facilities now as when the resources of South Kensington were at our disposal, and that an experimental demonstration may consequently involve more trouble and, perhaps, some additional cost. But with regard to the latter objection I have very little hesitation in saying, though I am quite without authority to do so, that, if necessary, payment out of the Society's funds of the reasonable expenses incidental to any appropriate demonstration would be readily sanctioned by the Council and approved by the Society generally. Let me remind Fellows that the exhibition of experiments or of apparatus is not necessarily accompanied by the reading of a formal paper, and that the experiments need not even be new, in the sense that they have never been exhibited elsewhere. Anything of general interest to physicists would always be heartily welcomed.

Among the objects for which the Physical and other kindred societies are established is a very important one to which no reference is made in the Memorandum of Association. One of the most useful functions of these institutions is to bring together and promote friendly intercourse among fellow-workers in a particular branch of science. In this connection I myself (if you will pardon me for referring to personal matters) owe a heavy debt of gratitude to the Physical Society. At the time when I was desirous of becoming a member, I was not personally known to a single man who was in a position to support my candidature. After some preliminary correspondence I introduced myself to Prof. Roberts-Austen, then one of the secretaries; he kindly gave me an introduction to Prof. Adams, the President, and the two were good enough to sign my recommendation form. Who furnished the third necessary signature I never ascertained. In spite of this somewhat inauspicious *début*, it was my good fortune after, and solely as the result of, a few years' more or less regular attendance at the meetings, to have made a large number of acquaintances, and, I may say, very good friends, among the leaders and workers in science. I have long regarded my connection with the Physical

Society as the source of one of the chief interests of my life; and for the highly valued honour you have done me in electing me to be your President I cannot sufficiently express my thanks.

It was mainly with the view of promoting social intercourse among the Fellows that the experiment was tried three years ago of providing tea before the meetings. I regret to say that the result was a distinct failure. For some reason or other the institution was never appreciated, and when it was found that, owing to the scanty attendance, the cost of a cup of tea had reached an exorbitant amount the luxury had to be abandoned.

I believe now that this want of success was the consequence of an oversight in the arrangements. In order to economise time Members of Council were served with tea in the room where they transacted the Society's business. This was a mistake. They ought to have gone upstairs with the rest of us and fulfilled the duties of hosts or rather, perhaps, of masters of the ceremonies. I hope that in the future the experiment may be repeated under the more favourable conditions which I have indicated.

I must refer to one other very efficient means of encouraging sociability among our members. Through the kindness of certain influential gentlemen belonging to the Society, we have from time to time been afforded the opportunity of holding a meeting in some well-known physical laboratory either in town or in the country. I myself have had the privilege of taking part in most agreeable pilgrimages to Oxford, Cambridge, and Bristol, and have also been among those who enjoyed the hospitality of Profs. Adams, Carey Foster, Ayrton, and Thompson at their laboratories in London. It gives me great pleasure to be able to say that the Council has accepted an invitation from our Fellow, Mr. Porter, to go to Eton for the next meeting, which will be held on Saturday, February 26th, and I trust that our appreciation of his kindness will be testified by a large gathering.

It is a little difficult to beg for favours, but I may be allowed to suggest to those who are in a position to exercise similar hospitality, that they have it in their power to contribute in a material degree and in more than a merely scientific sense to the wellbeing of the Society.

And now I pass to the second of my proposed topics.

Perhaps the most remarkable event of purely physical interest which has attracted attention during the past year is the discovery of the Zeeman effect. Towards the end of 1896 Dr. Zeeman of Amsterdam made the observation that the spectral lines characteristic of certain elements in a state of vapour, became broadened

if the vapours were submitted to the influence of a strong magnetic field. Later experiments showed that the lines, instead of being merely increased in breadth, could be actually divided. A single line, if seen in the direction of the magnetic force, became in many cases split into two, which were circularly polarised in opposite senses; while, if looked at across the lines of force, it was tripled, the two outer lines being plane-polarised in one direction and the middle line in the direction at right angles. The phenomenon has been shown to be consistent with and confirmatory of the hypothesis that the ions carry electrical charges. Exceedingly beautiful photographs of the Zeeman effect in zinc, cadmium, and iron, made by Mr. Thomas Preston, were exhibited by Prof. Fitzgerald at our last meeting.

Among the most important of the papers which have been published by our own Society is Dr. Chree's memoir on the Applications of Physics and Mathematics to Seismology, which ran through three numbers of our 'Proceedings.' The author has discussed from a theoretical standpoint the influences exerted by such agencies as ocean and estuary tides, changes of barometric pressure, and the gravitational action of the moon and the sun, in producing temporary distortion at the earth's surface. He has also given numerical examples and indicated the most favourable conditions for the direct observation of some of the effects.

Another paper of high scientific interest and possible technical value is that on the Magnetic Properties and Electrical Resistance of Iron as dependent upon Temperature, by Dr. David Morris. Dr. Morris's results may constitute an important contribution towards the physical data required for the solution of the mystery presented by the magnetic metals.

A paper representing a large amount of careful experiment, on the Condensation and Critical Phenomena of some Substances and Mixtures, was communicated by Dr. Kuenen.

Mention should also be made of Mr. Vincent's skilful work in connection with the instantaneous photography of mercury ripples. Several of the beautiful pictures which were exhibited upon the screen when his paper was read have been successfully reproduced in the July number of the 'Proceedings.'

A matter of the greatest interest to all physicists is the appointment of the National Physical Laboratory Committee, with Lord Rayleigh as Chairman. The terms of reference are as follows:—"To consider and report upon the desirability of establishing a National Physical Laboratory for the testing and verification of

instruments for physical investigation: for the construction and preservation of standards of measurement: and for the systematic determination of physical constant and numerical data, useful for scientific and industrial purposes. And to report whether the work of such an institution, if established, could be associated with any testing or standardising work already performed, wholly or partly at the public cost."

Prof. Silvanus Thompson and myself were appointed by the Council to give evidence before the Committee on behalf of the Physical Society.

The importance of accurate measurement in scientific research is sufficiently obvious. The value of much experimental work would be increased, much that is now practically impossible would become easy, and much valuable time would be saved if investigators could send their measuring instruments to be tested and certified by an undoubted authority.

Again, the important work of determining the physical properties or constants of matter should not be left to the caprice of volunteers. Such work is often of a tedious and uninteresting character; if it is to be of value, expensive apparatus is needed, and the operator must in many cases possess skill of a special kind; hence it is not readily undertaken unless there is some immediate object in view or some commercial purpose to be served.

The consequence is that our available tables of physical constants are of very unequal value. While in some parts they are excellent, in others they are seriously defective. The numbers given by various authorities often differ by as much as 10 per cent., and there are many gaps still remaining which ought to be filled up.

The work of enlarging and revising such tables should undoubtedly be undertaken in a systematic way by a national institution. No attempt should be made to render it directly remunerative, and little, if any, regard should be had to the possible utility of the results; it is impossible to foresee their value. Much of this work might perhaps be advantageously carried out in cooperation with similar institutions on the Continent.

The germ of such a laboratory as is desired already exists at Kew, and a better site could not easily be found. It has the advantage of being within easy reach of London and at the same time is almost entirely free from disturbances due to heavy traffic and the commercial applications of electricity.

There is reason to hope that this urgent national want will at

last receive at the hands of the Government the attention which it has so long deserved.

An event of last year which may, perhaps, be regarded by our Fellows with mixed feelings, is the foundation of a Röntgen Society under the presidency of Prof. Silvanus Thompson. The new Society has no doubt a wide field open to it in relation to matters physiological and medical and others which could not properly be included within the domain of Physics, and in such directions everyone must wish it a prosperous career. But if any investigator—even Prof. Thompson himself—should in the future make a notable discovery as regards the physical nature and properties of X-rays, I venture to express the hope that he will consider our own Society to be the proper recipient of his communication.

The publication by our Society of the highly-appreciated Abstracts was continued in 1897 upon the same lines as in the two preceding years, the papers abstracted being exclusively such as had been published in foreign journals and were of primarily scientific interest. In the present year, as the result of an arrangement with the Institution of Electrical Engineers, the number of the abstracts is to be greatly increased, British publications and papers of a technical character being included in their scope.

The arrangement in question is open to the objection that it entails the loss of our monopoly in the publication, for members of the Institution of Electrical Engineers will, like ourselves, receive copies of the Abstracts and will share with us whatever credit attaches to their production. The objection, however, appears to be in the main only a sentimental one and of small weight in relation to the substantial advantage accruing to our members—an advantage which could not possibly have been provided out of our own unaided resources.

I am happy to be able to announce that the work will still be carried on under the able editorship of Mr. Swinburne, with the assistance of a committee consisting of members of the two Societies interested. The Physical Society is under great obligations to Mr. Swinburne for the care and skill which he has for the last three years devoted to a laborious task, and to which is largely due the high reputation at present enjoyed by our Journal. There can, I think, be no reasonable doubt that under his guidance our latest venture will conduce to the continued welfare and prosperity of the Physical Society.

The Treasurer in Account with the Physical Society, from January 1st, 1897, to December 31st, 1897.

Dr.	£	s.	d.	£	s.	d.
Jan. 1, 1897. Balance in the Bank				154	6	3
Entrance-Fees	11	0	0			
Subscriptions for 1896	3	0	0			
„ 1897	9	0	0			
„ 1897	310	16	0			
„ 1898	2	2	0			
				335	18	0
Extra Subscriptions				32	10	0
Dividends				81	12	8
Grant from the British Association	100	0	0			
„ „ Royal Society	50	0	0			
				150	0	0
Sales				61	8	3
Advertisements				9	10	10
Return of Income Tax				7	1	0
Cheque passed through Treasurer's private account				16	0	0
Due to Treasurer				2	17	2
				£851	4	2

Cr.	£	s.	d.	£	s.	d.
Jan. 1, 1897. Balance paid to the Treasurer				3	4	10
Abstracts:—						
Editor	50	0	0			
„ Petty Cash	3	3	8			
Abstractors	72	11	7			
				125	15	3
Williams and Norgate (Periodicals)				18	10	10
Name Plate				2	2	0
Chemical Society				20	16	6
Attendance				2	5	0
Binding				2	6	6
Reporting				22	18	7
Bank Commission on Cheques				0	3	0
Taylor and Francis:—						
Proceedings	175	12	6			
Abstracts	215	12	6			
Miscellaneous	59	3	3			
Separate Copies	13	6	0			
				463	14	3
Petty Cash:—						
Mr. Elder	5	4	1			
Mr. Blakesley		12	1			
Dr. Atkinson	2	5	1			
				8	1	3
Deposit Account				21	1	0
Balance in the Bank, Dec. 31, 1897				160	5	2
				£851	4	2

EDMUND ATKINSON, *Treasurer.*

Audited and found correct.

ALFRED W. PORTER.
W. R. PIDGEON. } *Auditors.*

February 8, 1898.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31, 1897.

Assets.	£	s.	d.	Liabilities.	£	s.	d.
Subscriptions due, estimated	16	0	0	Subscription for 1898 in advance	2	2	0
£400 Furness 4 per cent. Debenture Stock	456	0	0	Balance due to the Treasurer	2	17	2
£1000 Midland 4 per cent. Preference Stock	1194	0	0	Abstracts, Oct.-Dec., 1897	29	16	0
£200 Metropolitan Board of Works $3\frac{1}{2}$ per cent. Stock	217	0	0	Printers	67	14	0
£270 Lancaster Corporation 3 per cent. Stock	227	6	8	Balance	2832	16	8
£254 2*s.* 9*d.* New South Wales $3\frac{1}{2}$ per cent. Inscribed Stock	249	7	5				
Balance in the Bank	160	5	2				
Guarantee Fund, on Deposit	415	6	7				
	£2935	5	10		£2935	5	10

EDMUND ATKINSON, *Treasurer.*

Audited and found correct.

ALFRED W. PORTER, } *Auditors.*
W. R. PIDGEON.

February 8, 1898.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1898-99.

February 26th, 1898.

Meeting held in the Physical Laboratory of Eton College by invitation of the Rev. T. C. PORTER.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society :—
Messrs. W. G. A. BOND and F. M. SAXELBY.

Mr. PORTER read the following communications :—

(1) Winter observations on the Shadow of the Peak of Tenerife, with a new method for measuring approximately the Diameter of the Earth.

(2) A new Theory of Geysers.

(3) A method of viewing Lantern Projections in Stereoscopic Relief.

March 11th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society :—
Messrs. J. M. DAVIDSON, C. V. DRYSDALE, and J. H. HOWELL.

Prof. J. D. Everett read a paper "On Dynamical Illustrations of certain Optical Phenomena."

Mr. R. A. Lehfeldt read a paper "On the Properties of Liquid Mixtures."

March 25th, 1898.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. H. Darwin, C. F. Green, and J. H. Thomson.

Mr. A. A. C. Swinton read a paper "On the Circulation of the Residual Gaseous Matter in a Crookes Tube."

Mr. A. Stansfield read a paper "On some Improvements in the Roberts-Austen Recording Pyrometer, and Notes on Thermoelectric Pyrometry."

April 22nd, 1898.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Prof. J. A. Ewing and Mr. S. G. Starling.

The Rev. T. C. Porter communicated a paper "On a Method of viewing Newton's Rings."

Prof. S. P. Thompson exhibited a model Triphase Generator and Motor."

May 13th, 1898.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. L. Gaster and J. Sampson.

Prof. Ayrton and Mr. Mather read a paper "On Galvanometers.—Part II."

May 27th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society:—
Messrs. F. W. CARTER and J. K. MOORE.

Mr. EDSER and Mr. BUTLER read a paper "On a Simple Interference Method of Reducing Prismatic Spectra."

Mr. A. A. C. SWINTON read a paper "On some further Experiments on the Circulation of the Residual Gaseous Matter in Crookes Tubes."

June 10th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

Prof. S. P. THOMPSON exhibited and described a Model illustrating Prof. Max Meyer's new Theory of Audition.

Mr. BARTON read a paper "On the Attenuation of Electric Waves along a Line of Negligible Leakage."

Mr. A. GRIFFITHS read a paper "On Diffusive Convection."

June 24th, 1898.

WALTER BAILY, Esq., Vice-President, in the Chair.

The following was elected a Fellow of the Society:—
Prof. EUGENIO SEMMOLA.

Prof. CARUS WILSON exhibited an Apparatus illustrating the action of Two Coupled Electric Motors.

Mr. J. QUICK exhibited Weedon's Expansion of Solids Apparatus.

Dr. F. G. DONNAN communicated a paper "On the Theory of the Hall Effect in a Binary Electrolyte."

October 28th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

Mr. W. R. PIDGEON read a paper on "An Influence Machine."

Prof. S. P. THOMPSON showed an Experiment on the Magneto-optic Phenomenon discovered by Righi.

Mr. A. CAMPBELL read a paper "On the Magnetic Fluxes in Meters and other Electrical Instruments."

November 11th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The discussion on Mr. CAMPBELL's paper read at the last meeting was continued.

Prof. W. B. MORTON communicated a paper "On the Propagation of Damped Electrical Oscillations along Parallel Wires."

November 25th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society:—
Messrs. J. B. BAYLIS, R. W. FORSYTH, and WILSON NOBLE.

Mr. R. A. LEHFELDT read a paper "On the Properties of Liquid Mixtures."

Mr. L. N. G. FILON read a paper "On certain Diffraction Fringes as applied to Micrometric Observations."

December 9th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society:—
Prof. H. L. CALLENDAR and Mr. R. S. WHIPPLE.

Dr. C. CHREE read a paper "On Longitudinal Vibrations in Solid and Hollow Cylinders."

Mr. ROSE-INNES and Prof. SYDNEY YOUNG read a paper "On the Thermal Properties of Normal Pentane."

January 27th, 1899.

G. GRIFFITH, Esq., Vice-President, in the Chair.

Dr. BARTON read a paper "On the Equivalent Resistance and Inductance of a Wire to an Oscillatory Discharge."

Mr. APPLEYARD exhibited a Dephlegmator and a Temperature Tell-tale.

Mr. LITTLEWOOD read a paper "On the Volume-changes accompanying Solution."

Annual General Meeting.

February 10th, 1899.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following Report of the Council was read by the Secretary:—

Since the last Annual General Meeting of the Society thirteen meetings have been held. One of these, on February 26th, 1898, took place in the Science Schools at Eton, on the kind invitation of the Rev. T. C. Porter, who provided some interesting demonstrations and made the Fellows of the Society welcome. All the other meetings were held in the rooms of the Chemical Society at Burlington House.

Twenty-one new Fellows have been elected into the Society; and the Council regrets to report that death has deprived the Society of a member of Council, Mr. Latimer Clark, and of Sir J. N. Douglass, Dr. John Hopkinson, Dr. J. E. Myers, the Rev. Bartholomew Price, Mr. Henry Perigal, and Dr. Eugen Obach. Short memoirs of some of these will be published in the Society's 'Proceedings.' There have been three resignations.

The Joint Committee of the Society and of the Institution of Electrical Engineers, which commenced its labours in January 1898, has published the first volume of 'Science Abstracts' in twelve monthly parts, being a continuation and expansion of the 'Abstracts of Papers on Physical Science' hitherto published by the Society alone. A much larger number of papers has been included, and the work has been very thoroughly done. Your Council is of opinion that it fulfils a want and is greatly appreciated. The expense has been considerable, and the estimate originally made has been somewhat exceeded. Your Council has, however, felt that this expense has been justified by the results, and has entered into arrangements for continuing the work of the Joint Committee.

It is confidently hoped that it will be possible to arrange for the permanent production of 'Science Abstracts' on a scale at least as extensive as the present.

The Report of the Council was adopted.

The Report of the Treasurer was read and adopted.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—Prof. Oliver J. Lodge, D.Sc., F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. Gladstone, F.R.S.; Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; Lord Kelvin, D.C.L., LL.D., F.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.; Prof. W. E. Ayrton, F.R.S.; Prof. G. F. Fitzgerald, M.A., F.R.S.; Prof. A. W. Rücker, M.A., F.R.S.; Capt. W. de W. Abney, R.E., C.B., D.C.L., F.R.S.; Shelford Bidwell, M.A., LL.B., F.R.S.

Vice-Presidents.—T. H. Blakesley, M.A.; C. Vernon Boys, F.R.S.; G. Griffith, M.A.; Prof. J. Perry, D.Sc., F.R.S.

Secretaries.—W. Watson, B.Sc.; H. M. Elder, M.A.

Foreign Secretary.—Prof. S. P. Thompson, D.Sc., F.R.S.

Treasurer.—Dr. E. Atkinson.

Librarian.—W. Watson, B.Sc.

Other Members of Council.—Prof. H. E. Armstrong, D.Sc., F.R.S.; Walter Baily, M.A.; R. E. Crompton; Prof. J. D. Everett, D.C.L., F.R.S.; Prof. A. Gray, LL.D., F.R.S.; E. H. Griffiths, M.A., F.R.S.; Prof. J. Viriamu Jones, M.A., F.R.S.; S. Lupton, M.A.; Prof. G. M. Minchin, M.A., F.R.S.; J. Walker, M.A.

Votes of thanks were passed to the Auditors, the Officers and Council, and to the Chemical Society.

The newly-elected President (Prof. Lodge) then took the Chair and delivered an Address, which is printed on pp. 343–386 of the current number of the 'Proceedings.'

The Treasurer in Account with the Physical Society, from January 1st, 1898, to December 31st. 1898.

Dr.	£	s.	d.	£	s.	d.
Jan. 1, 1897. Balance in the Bank				160	5	2
Entrance-Fees	21	0	0			
Subscriptions for 1897	4	4	0			
" 1898	346	5	0			
Extra Subscriptions for 1898	33	12	0			
Subscriptions for 1899	2	2	0			
				407	3	0
Life Compositions				63	0	0
Completion of Voluntary Composition	5	15	0			
Arrears of incomplete Subscriptions	2	4	0			
				7	19	0
Dividends				77	6	8
Sales of publications				48	3	0
Grant from the British Association	100	0	0			
" " Institution of Civil Engineers	100	0	0			
				200	0	0
Balance due to the Treasurer				13	14	5
				£977	11	3

Cr.	£	s.	d.	£	s.	d.
Balance due to the Treasurer, Jan. 1, 1898				2	17	2
Abstractors for 1897				29	16	0
Taylor and Francis, for printing:—						
Abstracts for 1897	36	5	6			
Proceedings	88	14	0			
Separate Copies	14	11	0			
Miscellaneous Printing	35	8	0			
				174	18	6
Asiatic Society, annual rent for name plate				2	2	0
Attendance				2	3	0
Binding				1	19	3
Cheque Book and Commission on Cheques					8	10
Reporting				22	17	5
Petty Cash:—						
Mr. Boys		12	7			
Mr. Elder	2	2	0			
Mr. Watson		15	0			
Dr. Atkinson	2	5	0			
				5	14	7
Mr. Swinburne: Science Abstracts	250	0	0			
Return of Grant to the Joint Committee	50	0	0			
" " "	50	0	0			
				350	0	0
Balance in the Bank				384	14	6
				£977	11	3

EDMUND ATKINSON, *Treasurer.*

Audited and found correct.

ALFRED W. PORTER, } *Honorary Auditors.*
WILLIAM BARLOW, }

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1898.

Assets.

	£	s.	d.	£	s.	d.
Subscriptions due, estimated				40	0	0
£400 Furness 4 per cent. Debenture Stock	456	0	0			
£1000 Midland 4 per cent. Preference Stock	1194	0	0			
£200 Metropolitan Board of Works $3\frac{1}{2}$ per cent. Stock	217	0	0			
£270 Lancaster Corporation 3 per cent. Stock	227	6	8			
£254 2*s.* 9*d.* New South Wales $3\frac{1}{2}$ per cent. Inscribed Stock	249	7	5			
				2343	14	1
Balance in the Bank				384	14	6
Guarantee Fund, on Deposit				424	12	2
				£3193	0	9

Liabilities.

	£	s.	d.
Balance due to the Treasurer	13	14	5
Rent due to the Chemical Society	20	9	0
Abstracts: due to the Joint Committee	70	0	0
Printers	43	14	3
Balance	3045	3	1
	£3193	0	9

EDMUND ATKINSON, *Treasurer.*

Securities examined and found correct.

ALFRED W. PORTER, } *Honorary Auditors.*
WILLIAM BARLOW, }

The Meeting then resolved itself into an Ordinary Science Meeting, and the following were elected Fellows of the Society:—

Messrs. David Woolacot and J. Donaldson.

Mr. B. Davis communicated a paper on "An Ampere-meter and a Voltmeter with a long Scale."

Dr. John Hopkinson, F.R.S., was born on July 27, 1849, and received his early scientific training at Owens College, Manchester, and Trinity College, Cambridge. He was Senior Wrangler and first Smith's Prizeman, as well as a Whitworth Scholar and a D.Sc. of London University. Shortly after leaving Cambridge he joined the firm of Messrs. Chance Brothers & Company, and while with them introduced many and important improvements in lighthouse illumination. He also conducted some important researches on the electrical properties of glass. In 1879 the first of the long series of papers in connection with electric lighting and engineering, which has rendered his name famous, was published. In 1886, in conjunction with his brother Edward Hopkinson, was published the paper on the general theory of the magnetic circuit, which had such a revolutionizing influence on the design of dynamo machinery.

He was a Fellow of the Royal Society, and was twice President of the Institution of Electrical Engineers. He became a Fellow of the Physical Society in 1878.

He was an expert climber, and on August 27, 1898, together with one of his sons and two daughters, he set out to climb the Petite Dent de Veisivi. As they did not return at nightfall, search parties were sent out, and the four bodies were found at the base of a cliff, having fallen from a height of about 500 feet. By this sad accident the scientific world lost a leader whose place cannot easily be filled.

Mr. Latimer Clark, F.R.S., was born at Great Marlow in 1822. He was engaged in a chemical industry in Dublin, when the rapid development of railways tempted him to serve as a pupil with his brother, Mr. Edwin Clark, on the Chester and Holyhead Railway. He assisted in the famous experiments which preceded the construction of the Britannia tubular bridge. The bridge was of so novel a kind that it could not be trusted to a Contractor, so that

from 1848 to 1850 Mr. Clark had an experience of experimenting, calculating, designing, and of directing workmen. His bent towards electrical experimenting caused him to be asked to join the staff of the Electric Telegraph Company, of which he was Engineer-in-chief from 1854 to 1861, and Consulting Engineer till 1870, when the Government took charge of telegraphic work. It is claimed for him that he was the first to suggest and carry out (in 1854) the pneumatic conveyance of parcels and letters. In 1854 he gave a lecture at the Royal Institution on the results of his experiments on submarine and subterranean electric conductors. In 1861 his paper read before the British Association introduced the idea of the use of definite practical electrical units and standards, with such names as Ohm, Farad, and Volt. In 1873 he brought before the Royal Society the results of his elaborate, costly, and long investigation which led to the use of the Clark's cell as our standard of Electromotive Force. He was a good practical astronomer; a photographer; a great collector of books on Electricity and Magnetism; a writer of a number of very useful handbooks; an inventor of many things now in common use; an engineer who constructed floating docks and large hydraulic contrivances; a manufacturer of electrical machinery and apparatus; a layer of submarine cables. He was President of the Institution of Electrical Engineers in 1875, and became a Fellow of the Royal Society in 1889. Mr. Clark was elected a Fellow of the Physical Society during its first Session, and at the time of his death was on the Council.

The Rev. Bartholomew Price, D.D., Master of Pembroke College, Oxford, died on December 29th, 1898, in the 81st year of his age.

He was widely known by his exhaustive treatises on the Calculus, and was more noted perhaps for his business and financial ability; he remained active and vigorous until he was eighty, and by his kindliness made for himself a large circle of friends.

Beyond this, however, there is a side to his career which is of special interest to our Society: at a very early date his sagacity was recognized at Oxford: his clear and temperate views gained for him such influence that on important occasions the leaders of all parties would seek his counsel, and what was to him right and politic did not generally meet with opposition. This power was exercised in the cause of liberal education. In all that has been done for the study of Natural Science at Oxford during the last

forty years no one has taken a greater part than the late Professor Price. Much still remains to be done, but the importance of his work may be seen, when it is realized at what point it was necessary to begin. The regulation, that some knowledge of Arithmetic must be shown by all students entering the University of Oxford, was proposed and carried through by our late Fellow. He was always working for progress, but was never ruffled by the opposition of the mediæval spirit: without his tact and good temper the progress of learning, as we understand it, might have been delayed for many years in Oxford.

It may be of interest to note that he was the President of Section A, at the famous meeting of the British Association at Oxford in 1860, when Bishop Wilberforce, in the one incautious moment of his life, attacked Professor Huxley.

Mr. Henry Perigal was elected a member of the Physical Society during the first Session of the Society. He died at the advanced age of ninety-seven years, and till within a short period of his death was a constant attendant at the meetings of the Society. He was Treasurer of the Royal Meteorological Society and a Fellow of the Royal Astronomical Society, and a member of several other scientific bodies.

Dr. Eugen F. A. Obach, F.I.C., F.C.S., was elected a Fellow of the Society in May 1880. He was a Doctor of Philosophy of Heidelberg. He was in the employ of Messrs. Siemens Bros. & Co., of Woolwich, and devoted himself to the study of guttapercha and indiarubber. He delivered the Cantor Lectures on these substances at the Society of Arts in 1898. He died at the early age of forty-six.

Sir James Nicholas Douglass, F.R.S., became a Fellow of the Society in 1889. He was many years Engineer-in-chief to the Hon. Corporation of Trinity House, and built several lighthouses. He also made several improvements in the illuminating apparatus for lighthouses and lightships.